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http ://freakonometrics.hypotheses.org/

Université de Rennes 1, January 2017

Welfare, Inequality & Poverty

References

This course will be on income distributions, and the econometrics of inequality and poverty indices. For more general thoughts on inequality, equality, fairness, etc., see

- Atkinson & Stiglitz Lectures in Public Economics, 1980
- Fleurbaey & Maniquet A Theory of Fairness and Social Welfare, 2011
- Kolm Justice and Equity, 1997
- Sen The Idea of Justice, 2009

(among others...)

References

For this very first part, references are

— Norton & Ariely Building a Better America—One Wealth Quintile at a Time, 2011 [Income]

— Atkinson & Morelli Chartbook of Econonic Inequality, 2014 [Comparisons]

— Piketty Capital in the Twenty-First Century, 2014 [Wealth]

— Guélaud, Le nombre de pauvres a augmenté de 440.000 en France en 2010, 2012 [Poverty]

— Burricand, Houdré & Seguin Les niveaux de vie en 2010

— Houdré, Missègue & Seguin Inégalités de niveau de vie et pauvreté, 2012

— Jank & Owens Inequality in the United States, 2013 [Welfare]

Those slides are inspired by Emmanuel Flachaire's Econ-473 slides, as well as Michel Lubrano's M2 notes.

Wealth Distribution, Perception vs. Reality

Norton & Ariely Building a Better America—One Wealth Quintile at a Time, 2011

Fig. 2. The actual United States wealth distribution plotted against the estimated and ideal distributions across all respondents. Because of their small percentage share of total wealth, both the "4th 20%" value (0.2%) and the "Bottom 20%" value (0.1%) are not visible in the "Actual" distribution.

data (Actual) from Wolf Recent Trends in Household Wealth, 2010.

Wealth Distribution, Perception vs. Reality

Norton & Ariely Building a Better America—One Wealth Quintile at a Time, 2011

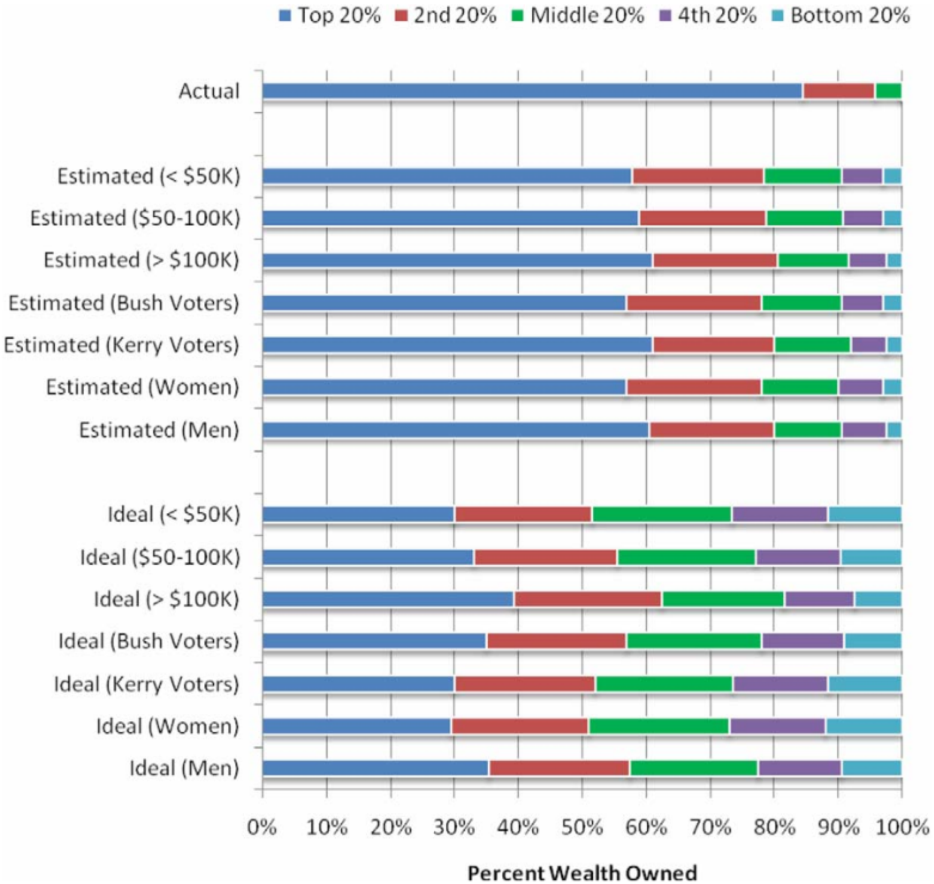
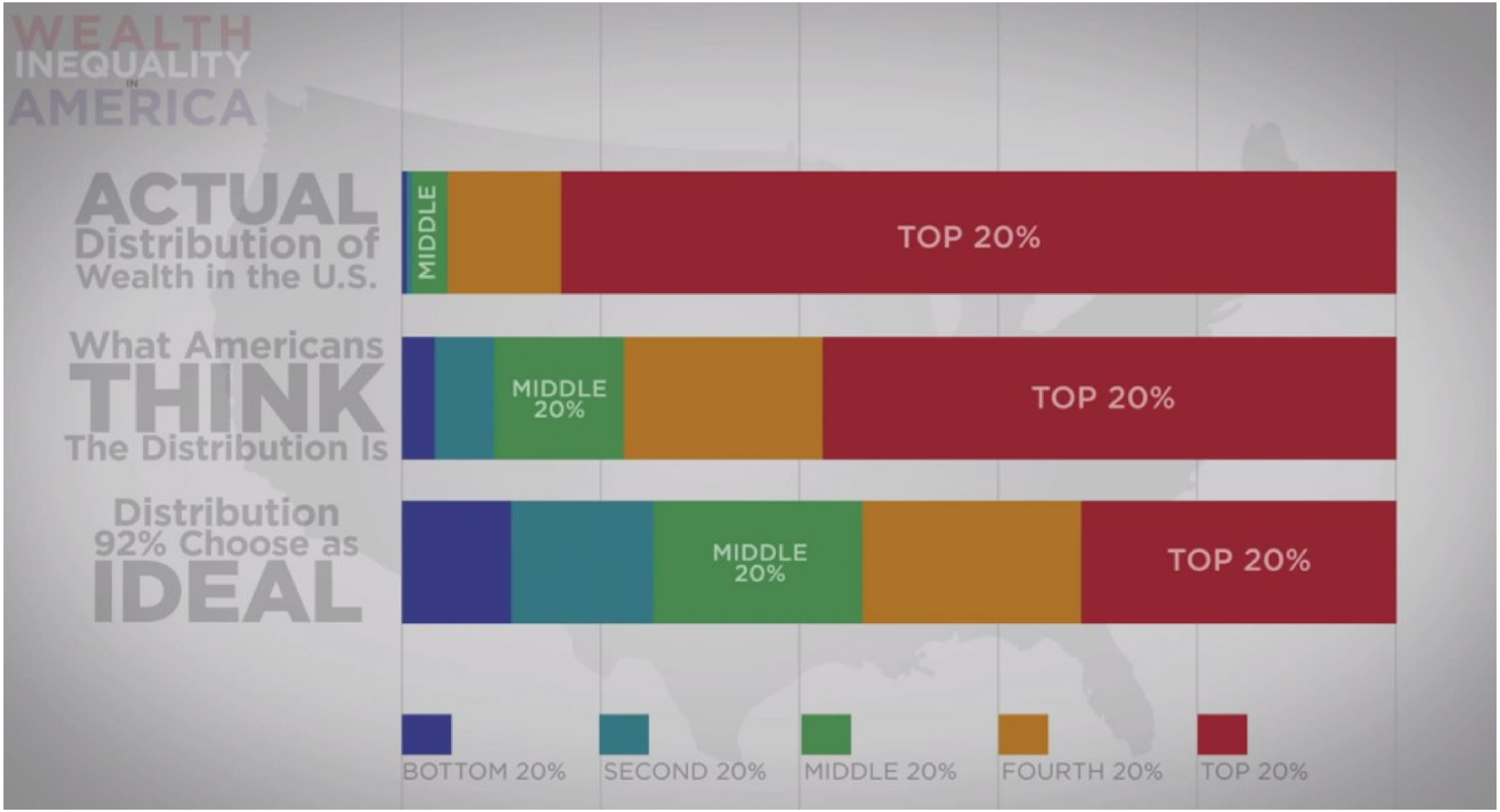


Fig. 3. The actual United States wealth distribution plotted against the estimated and ideal distributions of respondents of different income levels, political affiliations, and genders. Because of their small percentage share of total wealth, both the "4th 20%" value (0.2%) and the "Bottom 20%" value (0.1%) are not visible in the "Actual" distribution.

Wealth Distribution, Perception vs. Reality

Watch https://www.youtube.com/watch?v=QPKKQnijnsM



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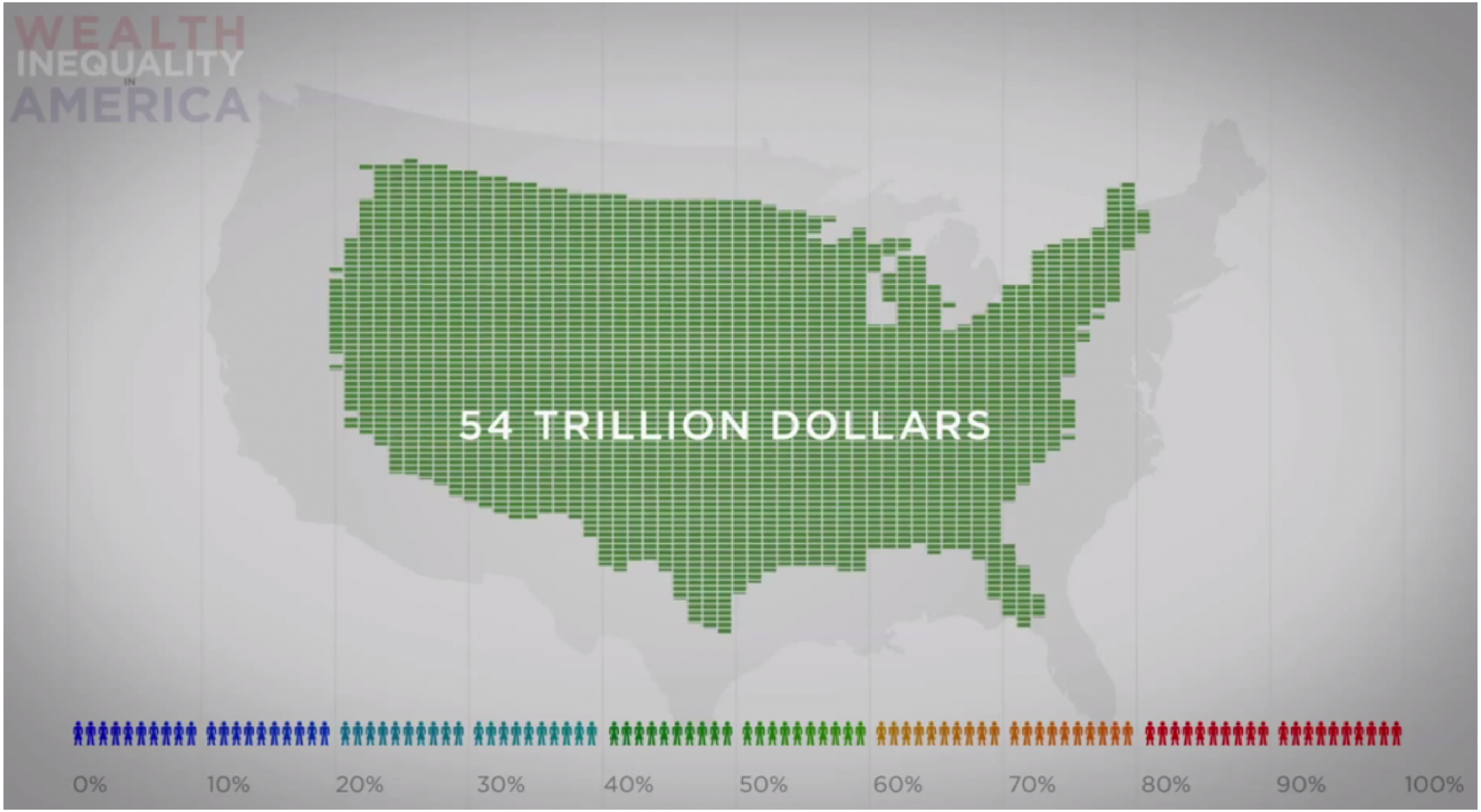

Wealth Distribution, Perception vs. Reality

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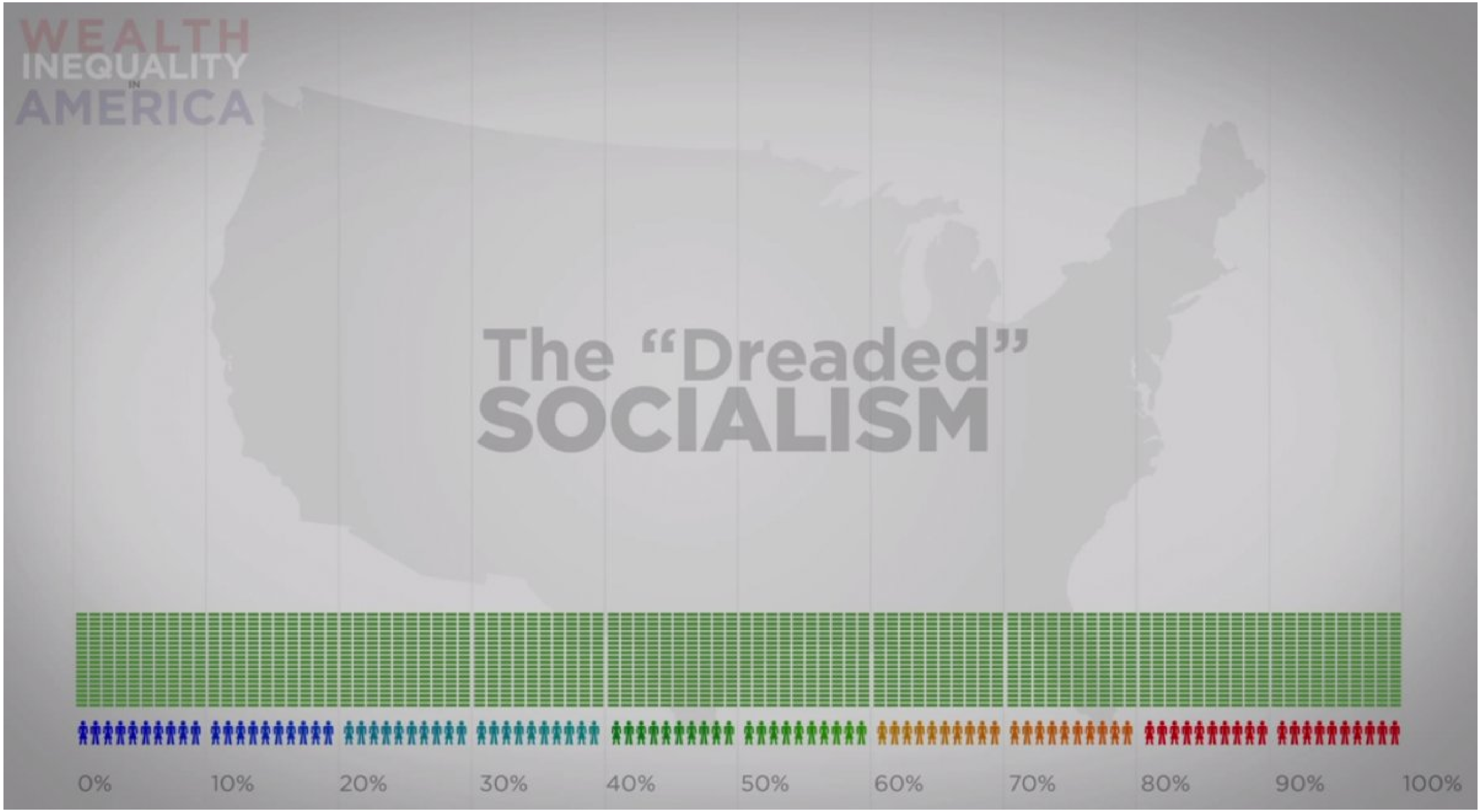

Wealth Distribution, Perception vs. Reality

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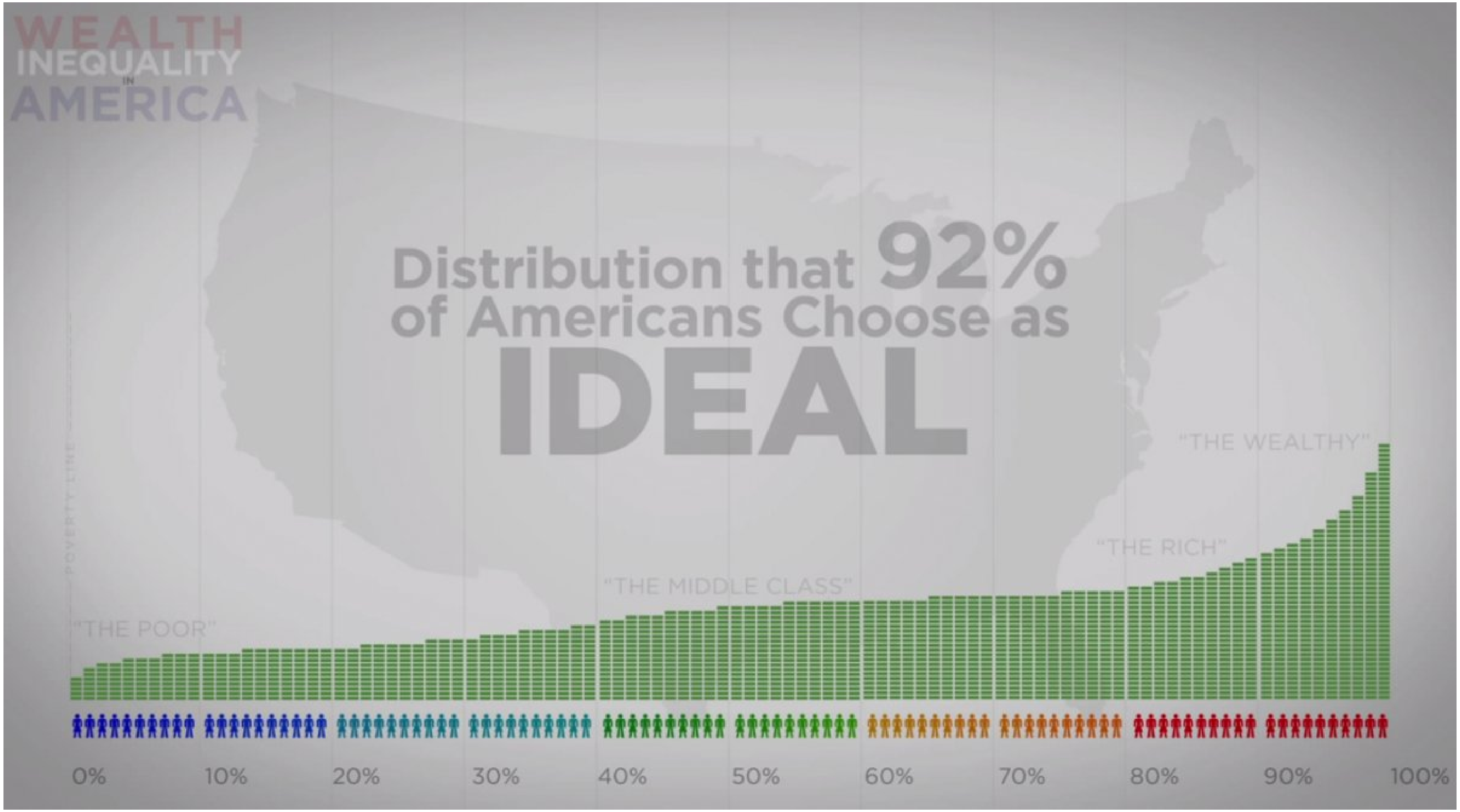

Wealth Distribution, Perception vs. Reality

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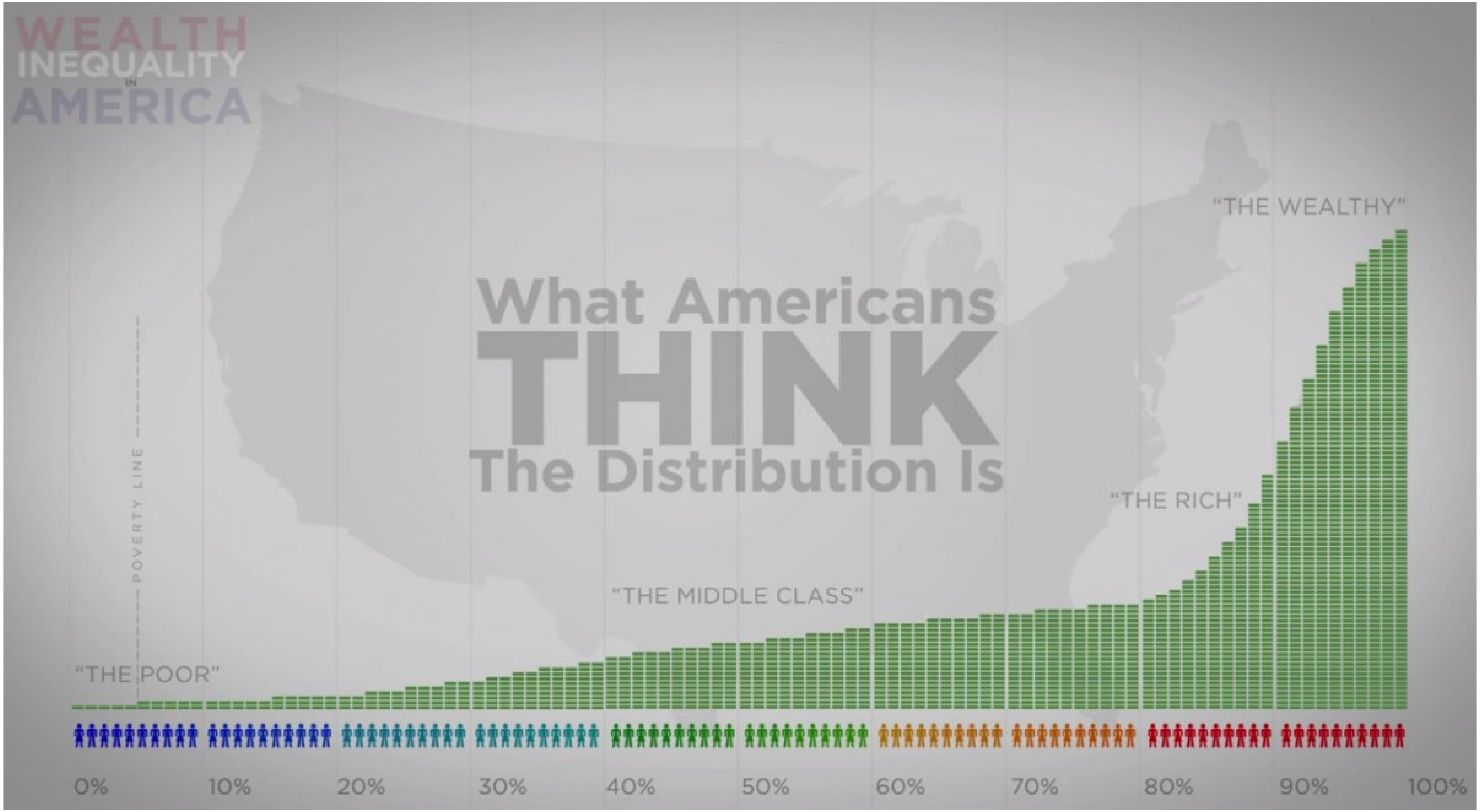

Wealth Distribution, Perception vs. Reality

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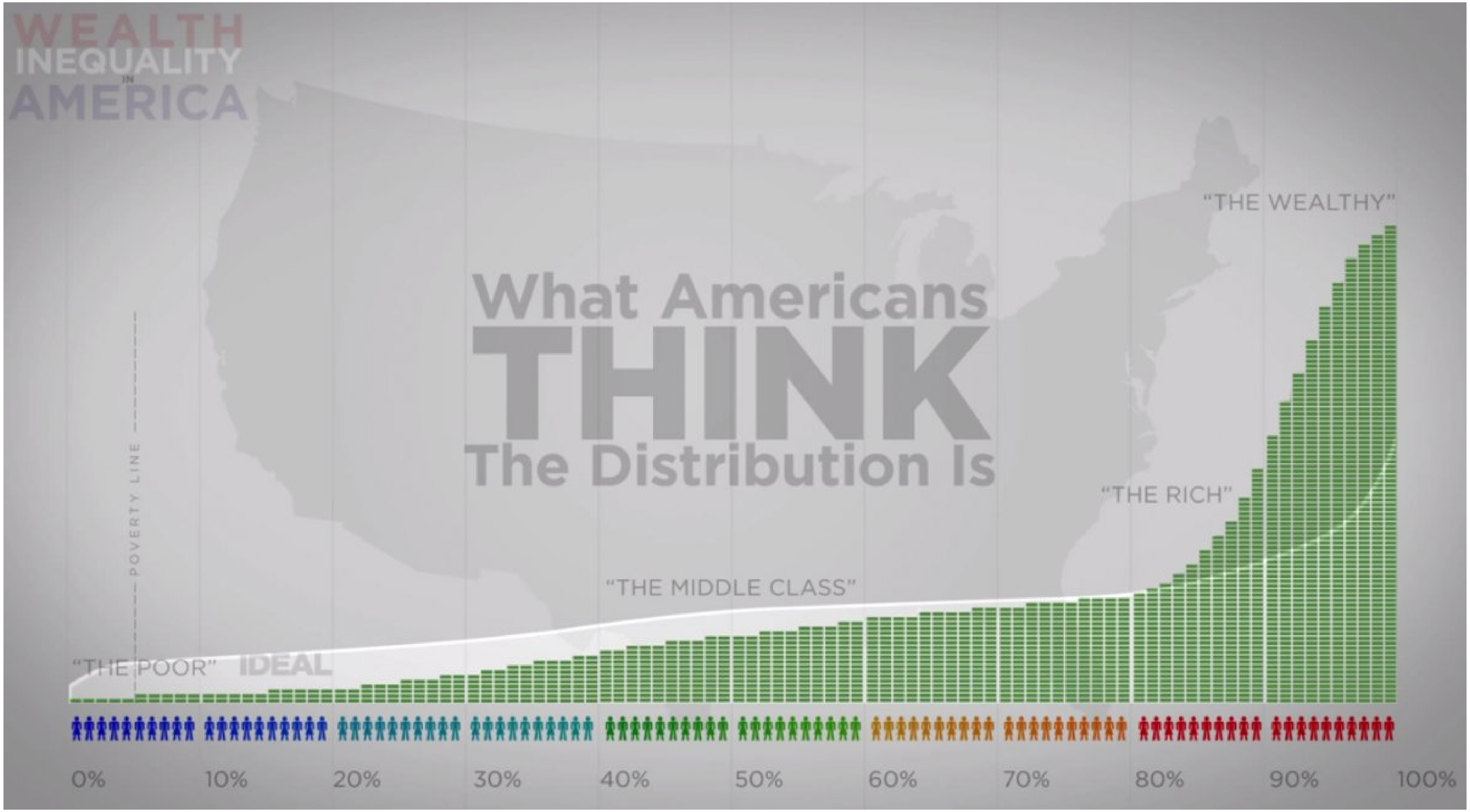

Wealth Distribution, Perception vs. Reality

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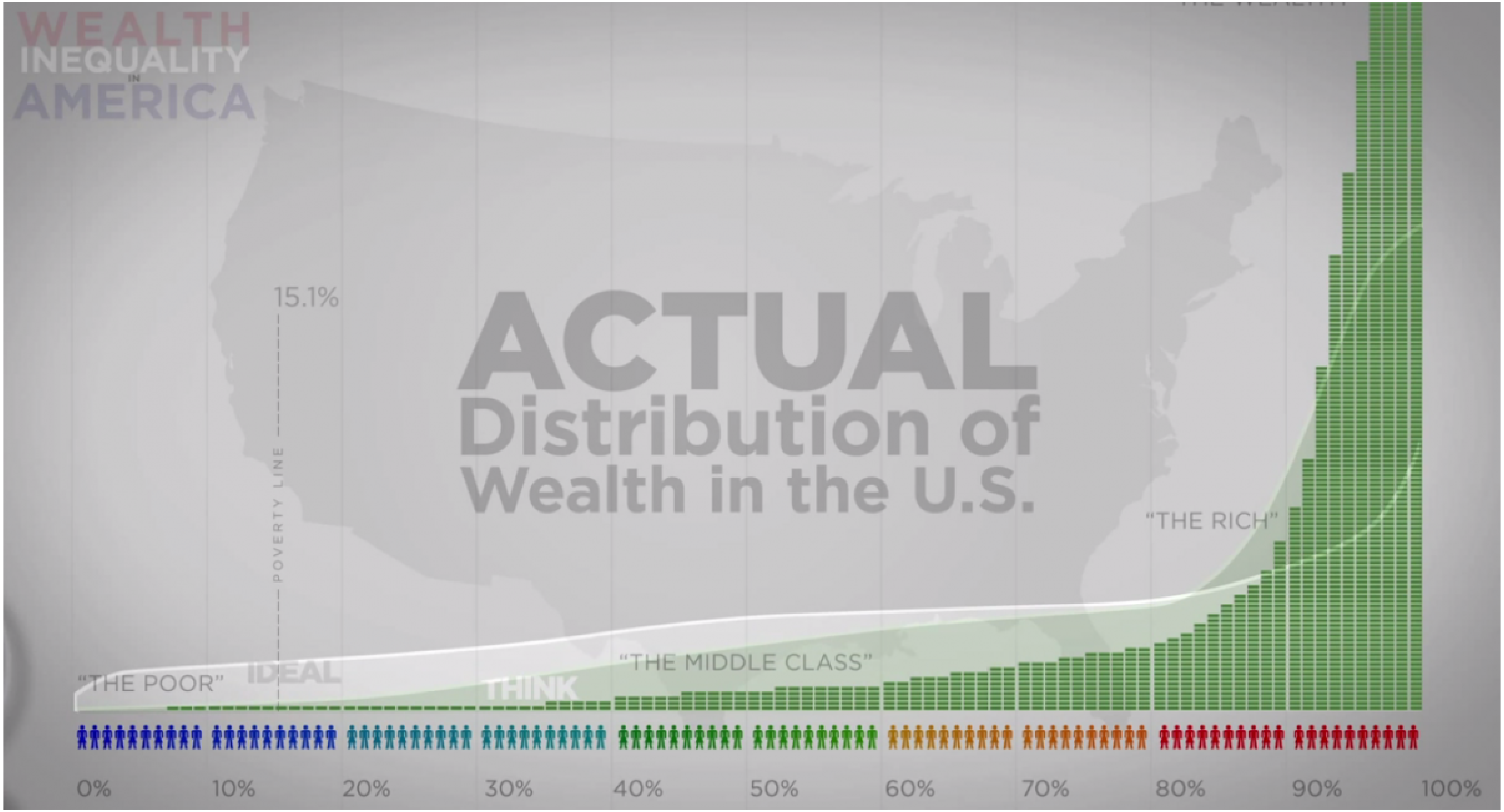


Comparing Inequalities in several countries

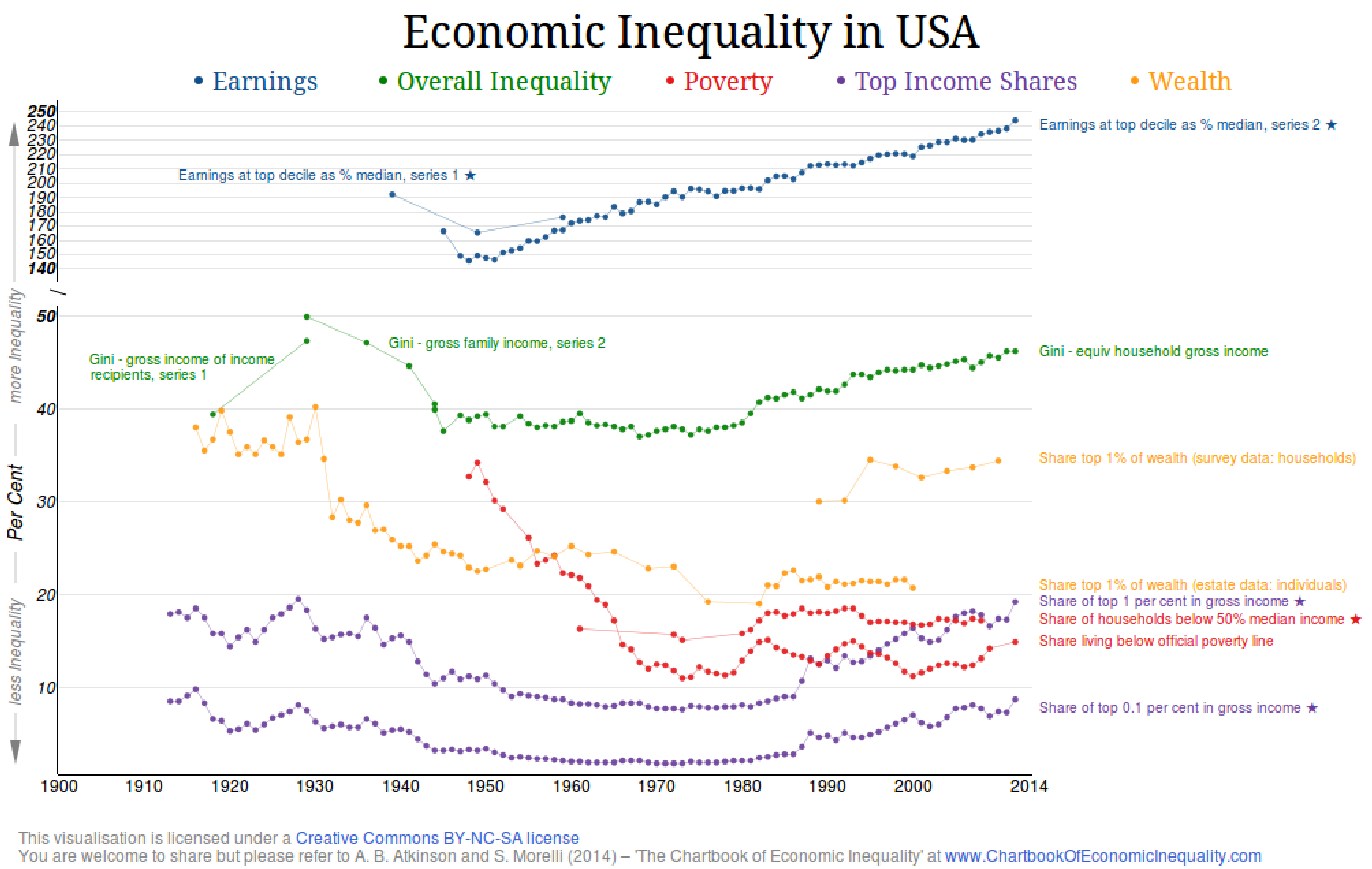
Atkinson & Morelli Chartbook of Econonic Inequality, 2014

in Argentina, Brazil, Australia, Canada, Finland, France, Germany, Ice- land, India, Indonesia, Italy, Japan, Malaysia, Mauritius, Netherlands, New Zealand, Norway, Portugal, Singapore, South Africa, Spain, Sweden, Switzerland, the UK and the US, five indicators covering on an annual basis :

- Overall income inequality ;
- Top income shares
- Income (or consumption) based poverty measures ;
- Dispersion of individual earnings ;
- Top wealth shares.

Comparing Inequalities in several countries

See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. U.S.A.



Comparing Inequalities in several countries

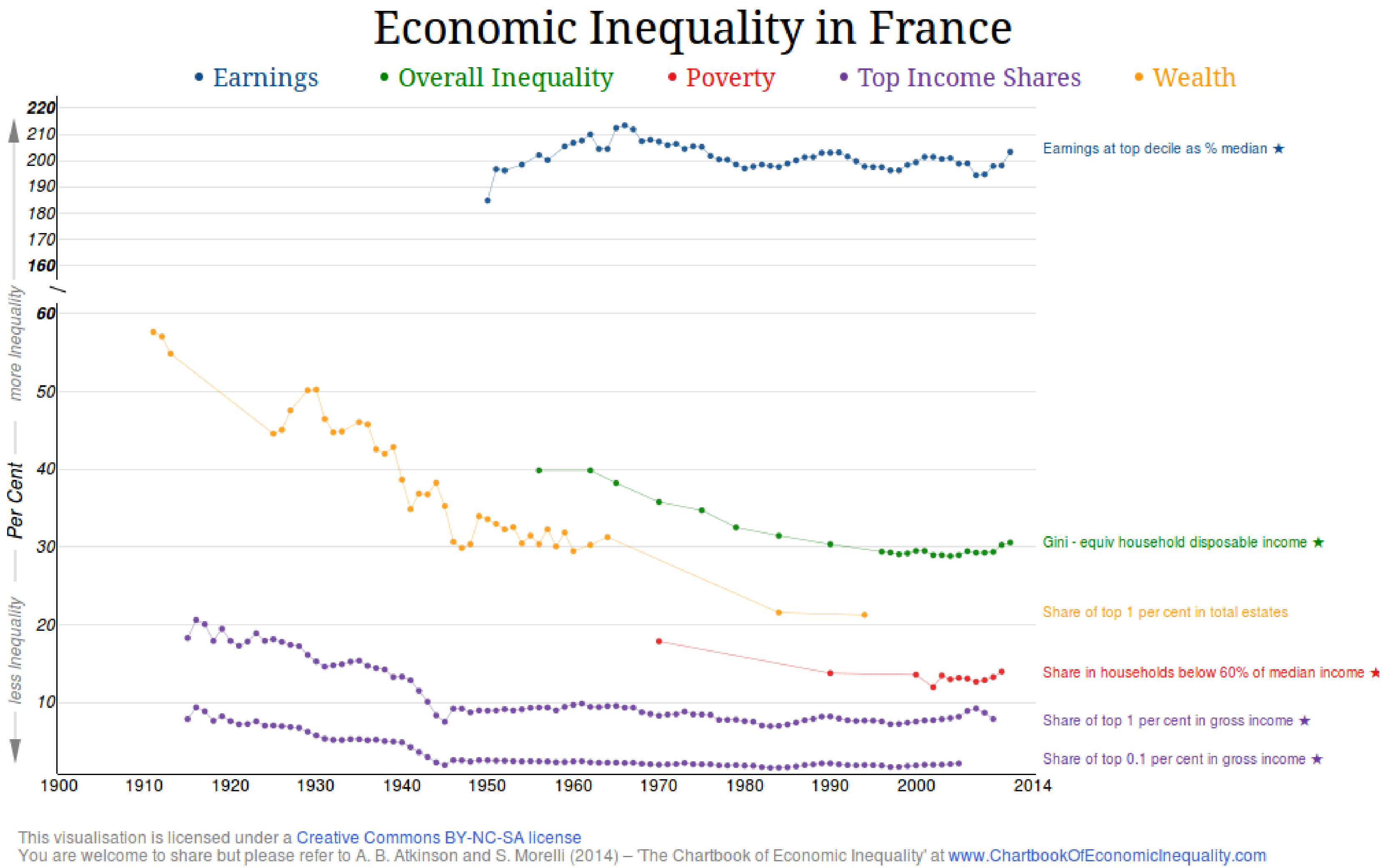
See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. U.S.A.

Overview of Trends of Economic Inequality in the USA

Has the dispersion of earnings been increasing in recent decades?	Yes, the top decile of earnings has risen from 150 per cent of median in 1950 to 244 per cent in 2012.
Has overall inequality increased in recent years?	Yes, the Gini coefficient for gross income now 7 percentage points higher than in 1980.
Have there been periods when overall inequality fell for a sustained period?	Yes, from 1929 to 1945.
Has poverty been falling or rising in recent decades?	Official poverty measure fell from 1948 to 1970s, since then cyclical variation about constant level.
Has there been a U-pattern for top income shares over time?	Yes, top gross income shares fell from 1928 to the 1970s; since mid-1970s have more than doubled.
Has the distribution of wealth followed the same pattern as income?	Top wealth shares generally decreased till 1982 but have not followed the upward trend in top incomes.
Additional noteworthy features	Earnings dispersion widened during the Period from 1950 to 1970 but overall income inequality did not increase.

Comparing Inequalities in several countries

See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. France



Comparing Inequalities in several countries

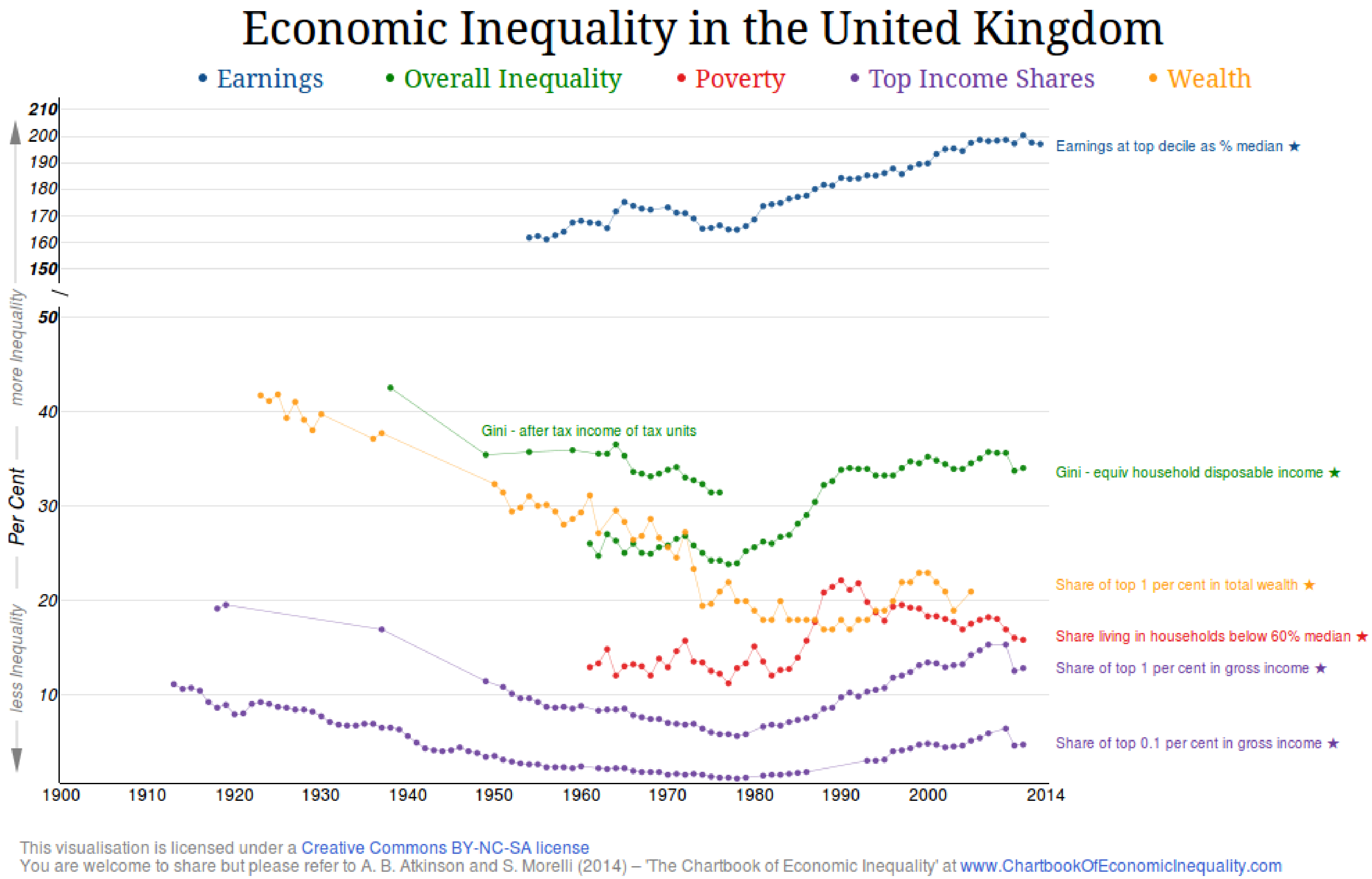
See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. France

Overview of Trends of Economic Inequality in France

Has the dispersion of earnings been increasing in recent decades?	No, earnings dispersion shows no apparent trend.
Has overall inequality increased in recent years?	No, Gini coefficient relatively stable since 1990s.
Have there been periods when overall inequality fell for a sustained period?	Yes, overall inequality (as well as wealth inequality and poverty) fell from the 1960s to the 1990s.
Has poverty been falling or rising in recent decades?	Fell from 1970 to 2000.
Has there been a U-pattern for top income shares over time?	No, top gross income shares fell from 1916 to 1945 and then stable over post-war period.
Has the distribution of wealth followed the same pattern as income?	Yes, top wealth share fell in post-war period while little change in top income shares.
Additional noteworthy features	Overall stability of inequality in recent years.

Comparing Inequalities in several countries

See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. U.K.



Comparing Inequalities in several countries

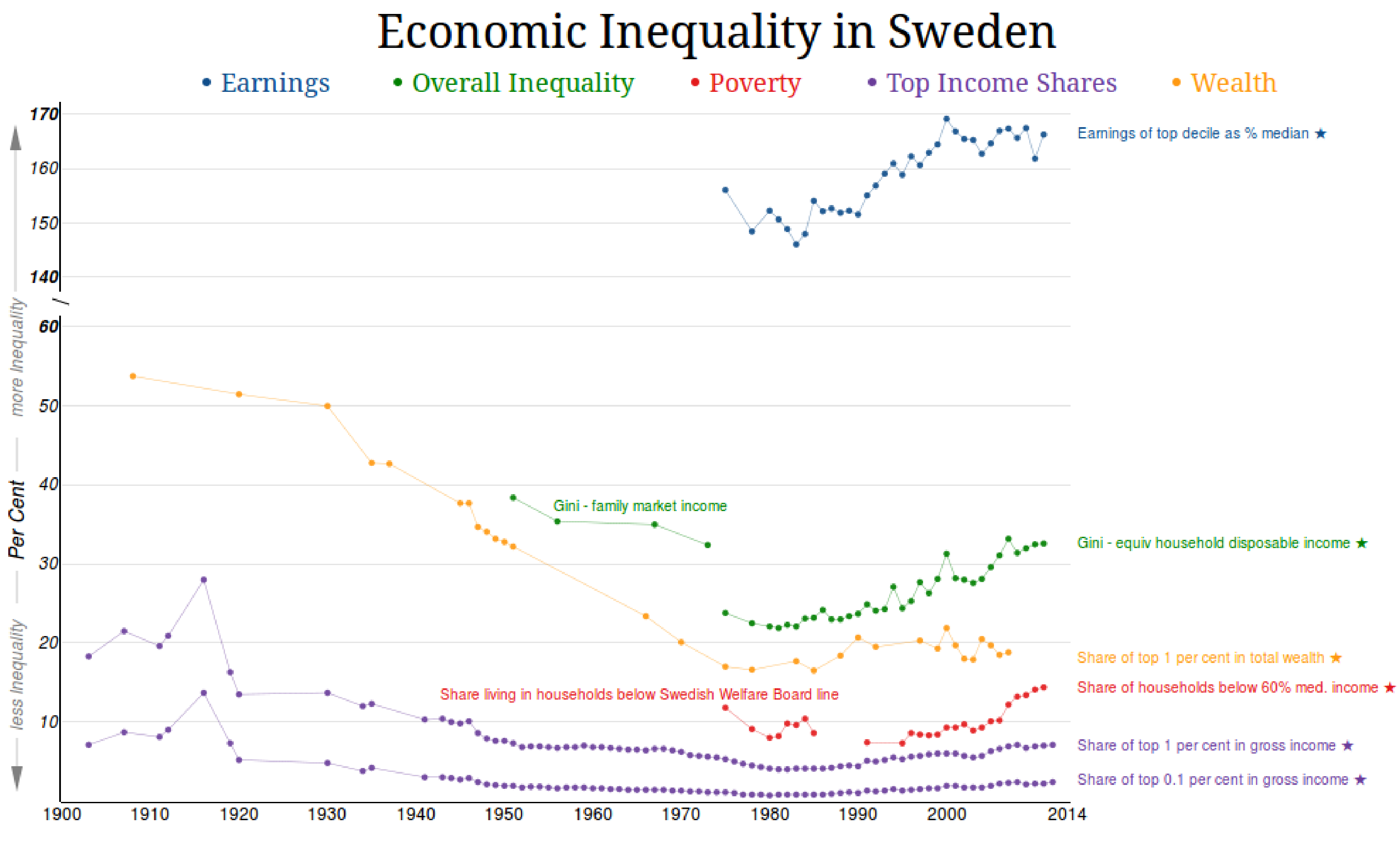
See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. U.K

Overview of Trends of Economic Inequality in the UK

Has the dispersion of earnings been increasing in recent decades?	Yes, the top decile of earnings has increased from 165 per cent of median in 1978 to 197 per cent in 2013.
Has overall inequality increased in recent years?	Yes, the Gini coefficient for equivalised disposable income is now around 10 percentage points higher than in 1980, but most of the increase took place in the 1980s.
Have there been periods when overall inequality fell for a sustained period?	Yes, during the Second World War and in late-1960s and 1970s.
Has poverty been falling or rising in recent decades?	Relative poverty rate in 1990 twice that in 1977; However, overall the poverty rate has been falling since the 1990s.
Has there been a U-pattern for top income shares over time?	Yes, top gross income shares fell from 1914 to the 1970s; since 1979 have more than doubled.
Has the distribution of wealth followed the same pattern as income?	Downward trend in top wealth shares from 1923 to end of 1980s; now levelled off.
Additional noteworthy features	Increase in income inequality and poverty in the 1980s proportionately much larger than increase in earnings dispersion. The top shares series have a break in 1990 (change in tax units from family to individual basis). The top income shares estimates for 2009-10 were affected by a significant bringing forward in that year in advance of the introduction of the 50 per cent top tax rate; the shares for the following years were correspondingly reduced.

Comparing Inequalities in several countries

See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. Sweden



Comparing Inequalities in several countries

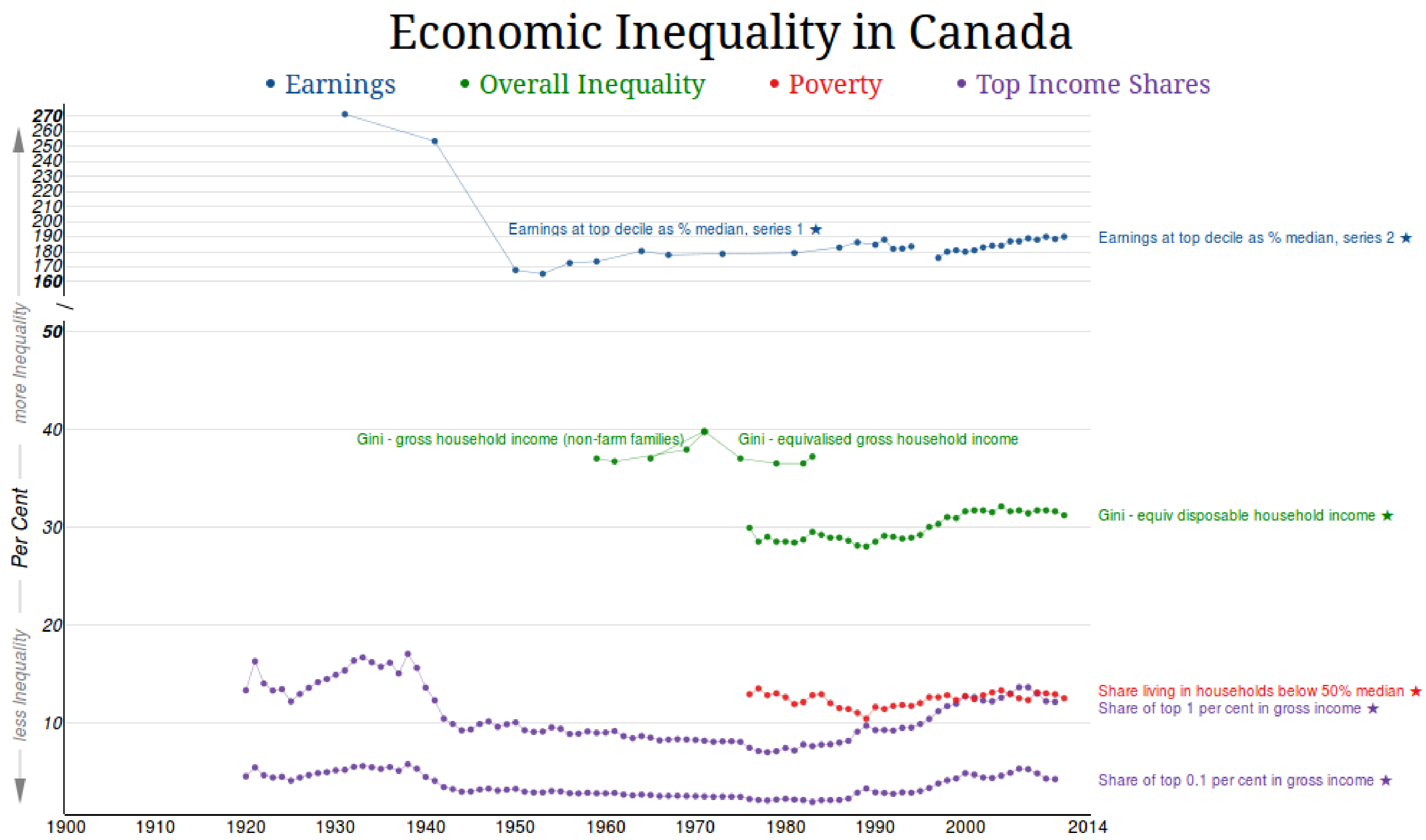
See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. Sweden

Overview of Trends of Economic Inequality in Sweden

Has the dispersion of earnings been increasing in recent decades?	Yes, the top decile of earnings has risen from 146 per cent of median in 1983 to 166 per cent in 2011.
Has overall inequality increased in recent years?	Yes, the Gini coefficient for equivalised disposable income is 10 percentage points higher in 2011 than in 1982.
Have there been periods when overall inequality fell for a sustained period?	Yes, much of twentieth century up to 1980s.
Has poverty been falling or rising in recent decades?	Rising. Relative poverty rate has doubled since 1995.
Has there been a U-pattern for top income shares over time?	Yes, top gross income shares fell from 1916 to 1980 and then rose.
Has the distribution of wealth followed the same pattern as income?	Similar till the end of 1980s. Top wealth shares show a downward trend from 1923 to end of 1980s; now levelled off.
Additional noteworthy features	Increase in overall inequality: during 1990s the average Gini was 25.5 while in the first decade of the twenty first century the average of Gini rose to 30. Top shares series have a break in 1971 (change in tax unit definition).

Comparing Inequalities in several countries

See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. Canada



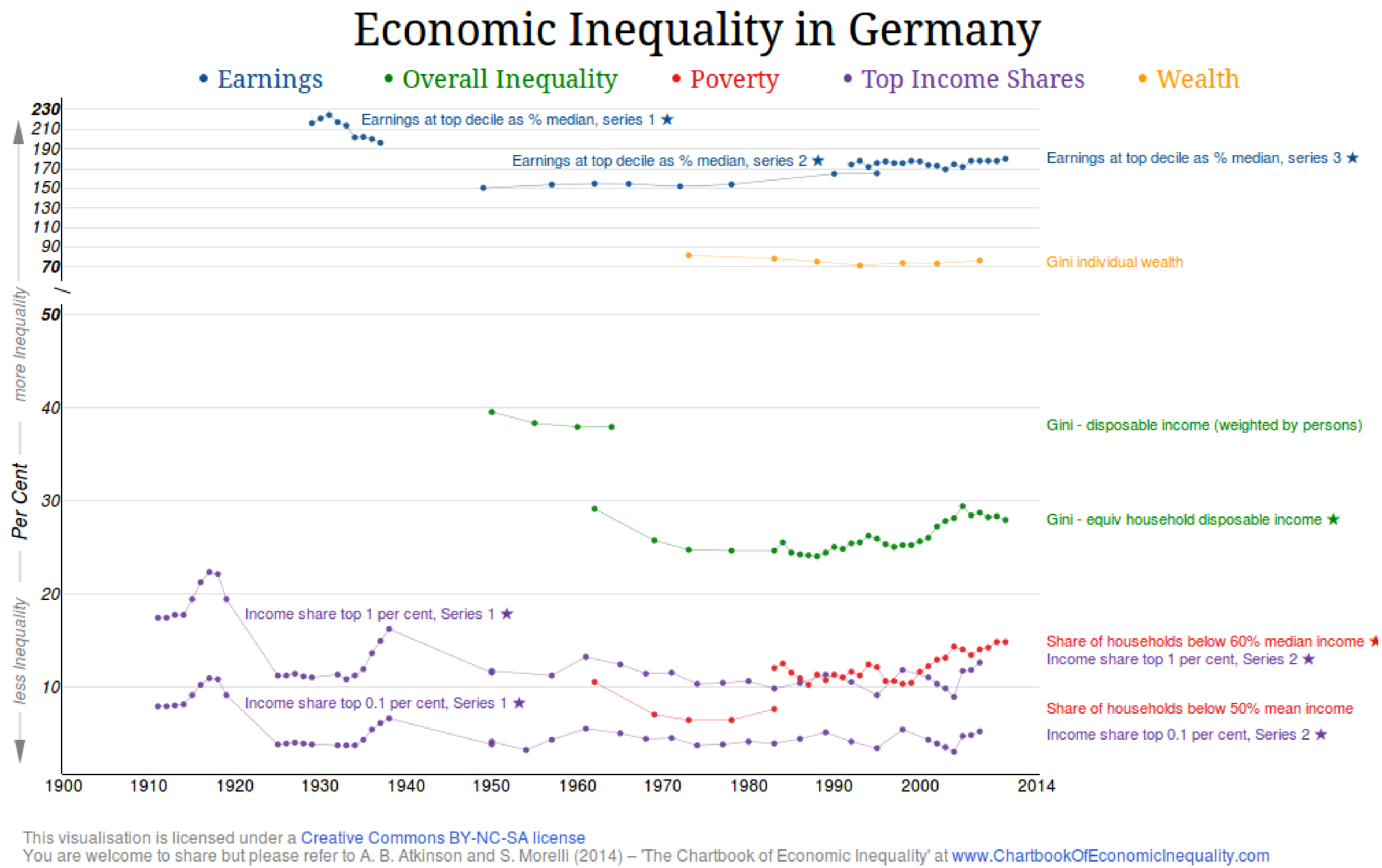
Comparing Inequalities in several countries

See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. Canada

Has the dispersion of earnings been increasing in recent decades?	Yes, top decile of earnings has been rising relative to the median since early 1950s.
Has overall inequality increased in recent years?	Yes, Gini coefficient is around 3 percentage points higher than in 1989 but most of the increase took place in the 1990s.
Have there been periods when overall inequality fell for a sustained period?	Incomplete evidence.
Has poverty been falling or rising in recent decades?	Poverty fell in the 1980's and then rose.
Has there been a U-pattern for top income shares over time?	Yes, top gross income shares fell from 1938 until the mid-1980s and then began to rise.
Has the distribution of wealth followed the same pattern as income?	No evidence.
Additional noteworthy features	

Comparing Inequalities in several countries

See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. Germany



Comparing Inequalities in several countries

See Atkinson & Morelli Chartbook of Econonic Inequality, 2014, e.g. Germany

Has the dispersion of earnings been increasing in recent decades?	Yes, top decile has risen from 150 per cent of median in 1950s to 190 per cent at end of 2000s.
Has overall inequality increased in recent years?	Yes, the Gini coefficient in 2010 was 3 percentage points higher than in 1998.
Have there been periods when overall inequality fell for a sustained period?	Overall inequality (and poverty) fell over the 1960s and 1970s.
Has poverty been falling or rising in recent decades?	Poverty rate increased from 10 per cent to 15 per cent between 1998 and 2010.
Has there been a U-pattern for top income shares over time?	No, top gross income shares were relatively stable over post-war period.
Has the distribution of wealth followed the same pattern as income?	Yes, Gini coefficient of individual wealth fell 10 percentage points from 1973 to 1993 and then began to rise.
Additional noteworthy features	

Comparing Inequalities in several countries

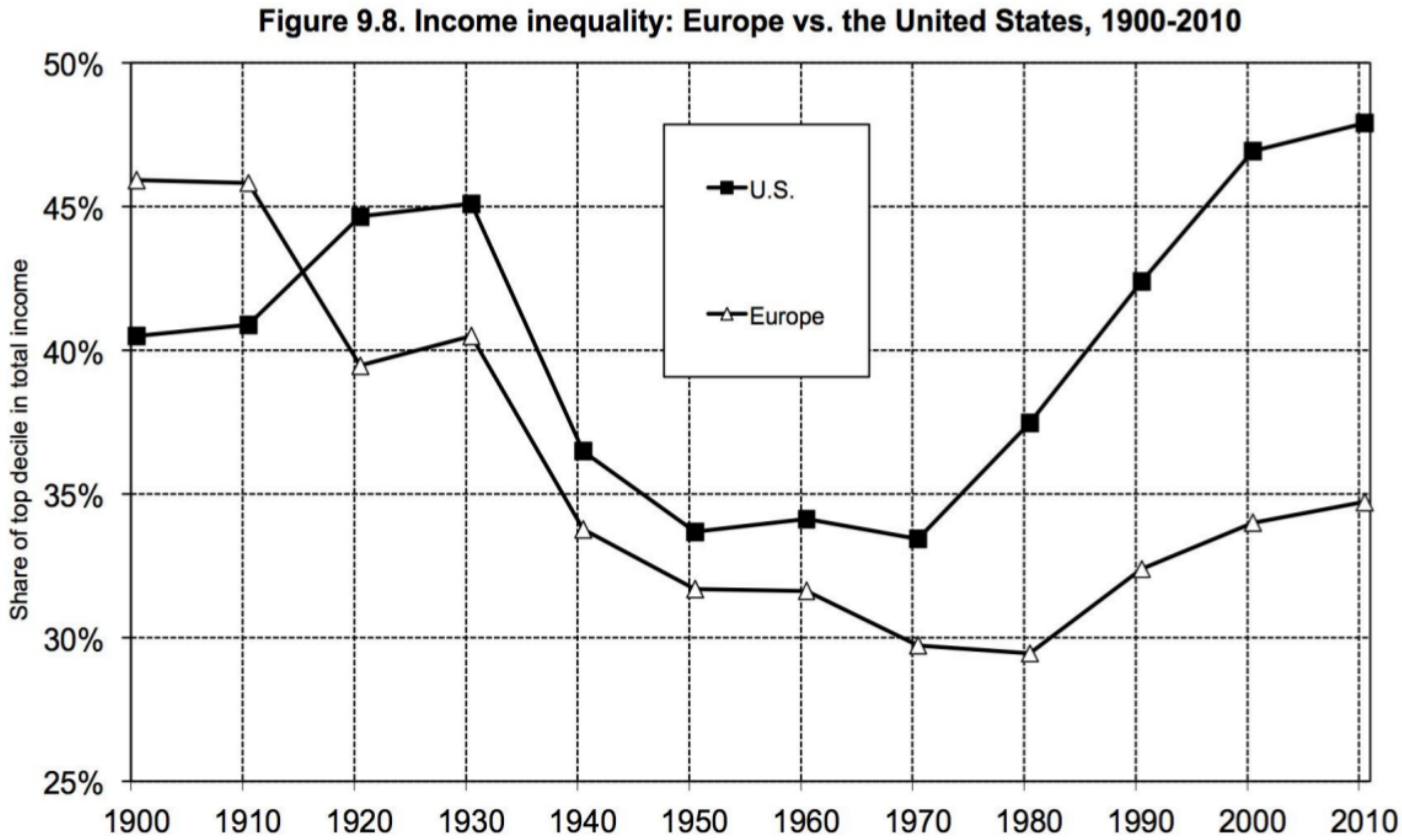
But one should be cautious about international comparisons,

- Inequality : Gini index based on gross income for U.S.A. and based on disposable income for Canada, France and U.K.
- Top income shares : Share of top 1 percent in gross income, for all countries
- Poverty : Share in households below 50% of median income for U.S.A. and Canada and below 60% of median income for France and U.K.

	USA	Canada	France	UK	Sweden	Germany
inequality	46.3	31.3	30.6	30.6	32.6	28.0
top income	19.3	12.2	7.9	7.9	7.1	12.7
poverty	17.3	12.6	14	14.0	14.4	14.9

Top Income Shares

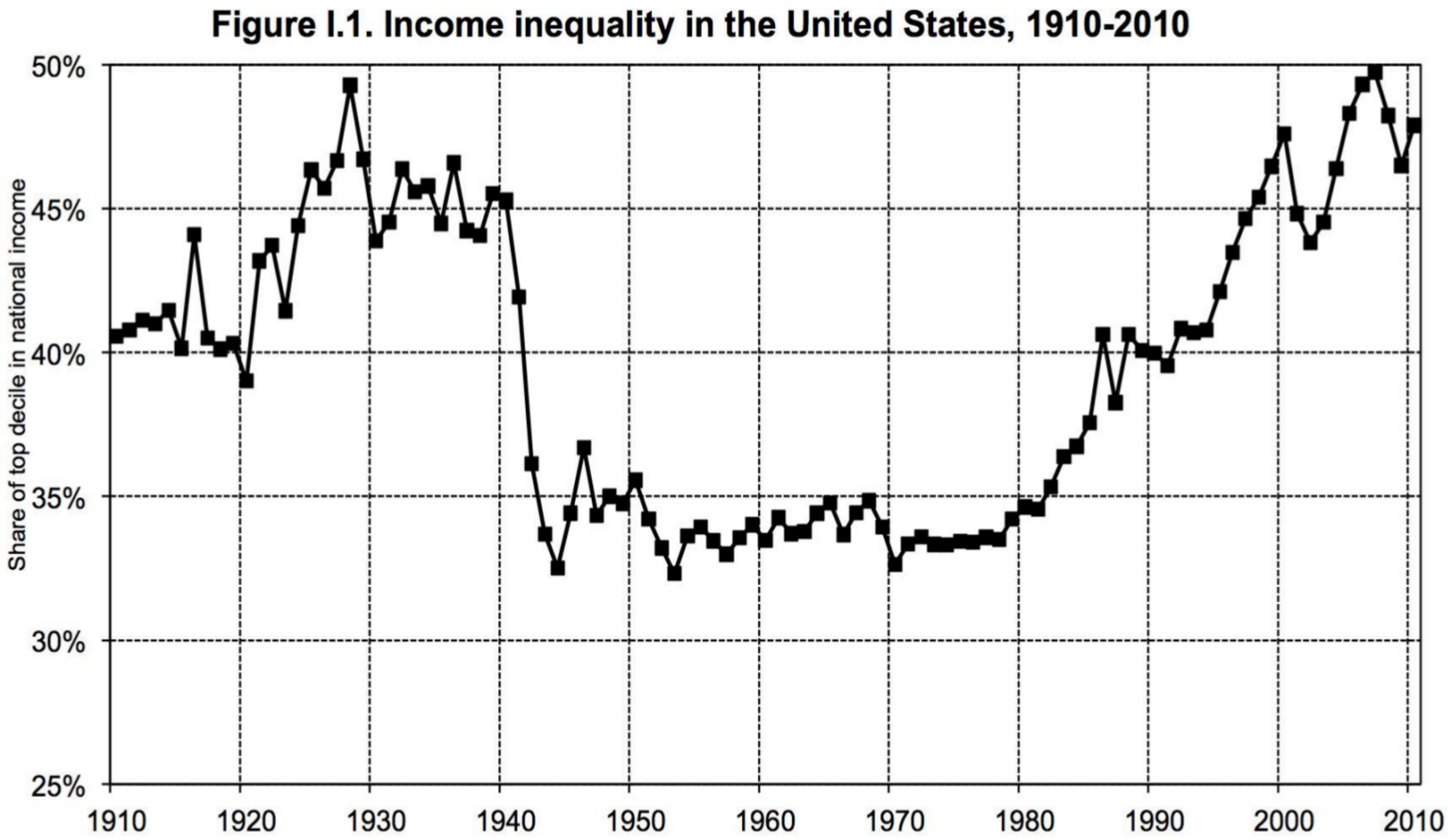
Piketty Capital in the Twenty-First Century, 2014



The top decile income share was higher in Europe than in the U.S. in 1900-1910; it is a lot higher in the U.S. in 2000-2010. Sources and series: see piketty.pse.ens.fr/capital21c.

Top Income Shares

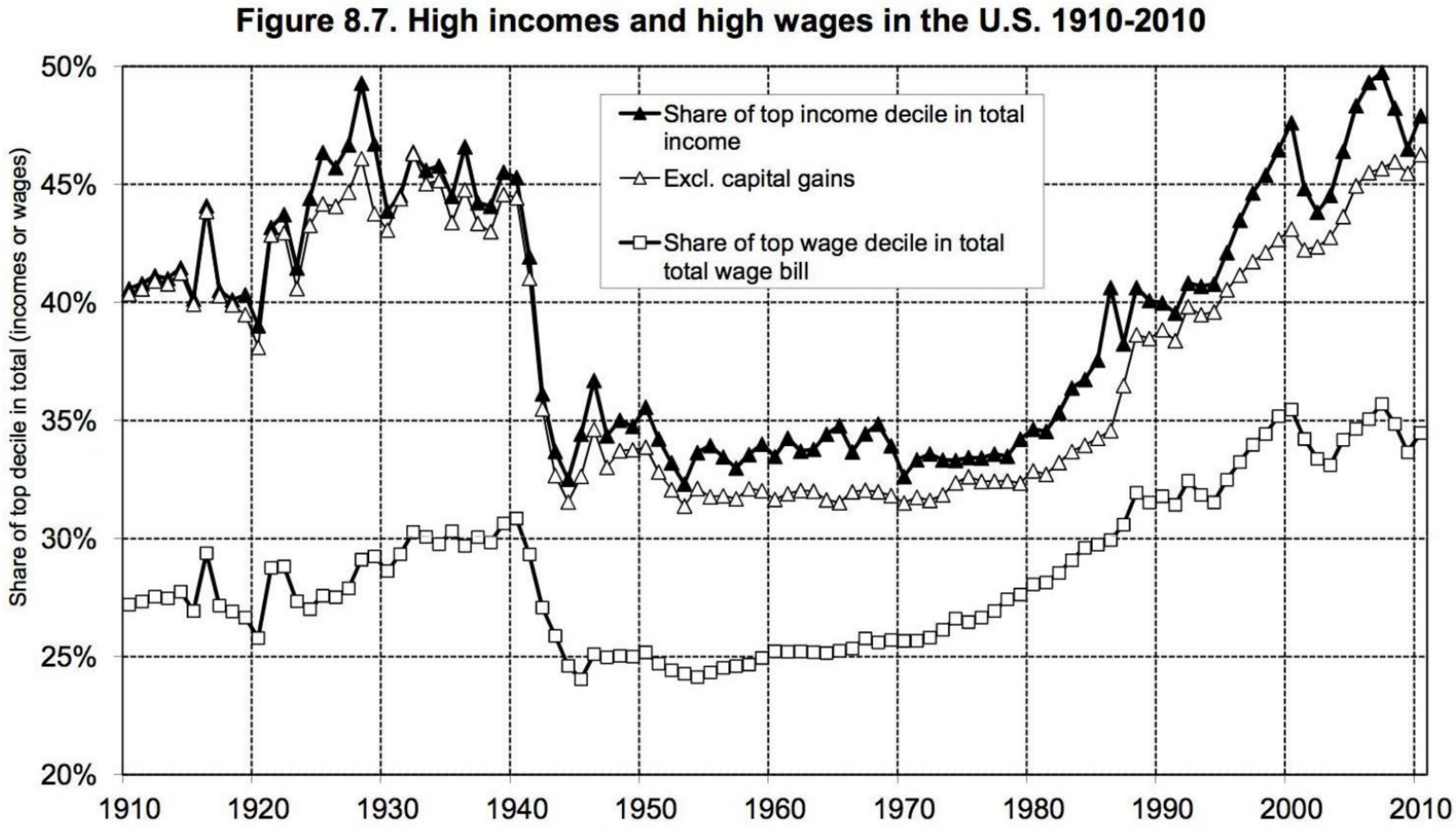
Piketty Capital in the Twenty-First Century, 2014



The top decile share in U.S. national income dropped from 45-50% in the 1910s-1920s to less than 35% in the 1950s (this is the fall documented by Kuznets); it then rose from less than 35% in the 1970s to 45-50% in the 2000s-2010s. Sources and series: see piketty.pse.ens.fr/capital21c.

Top Income Shares

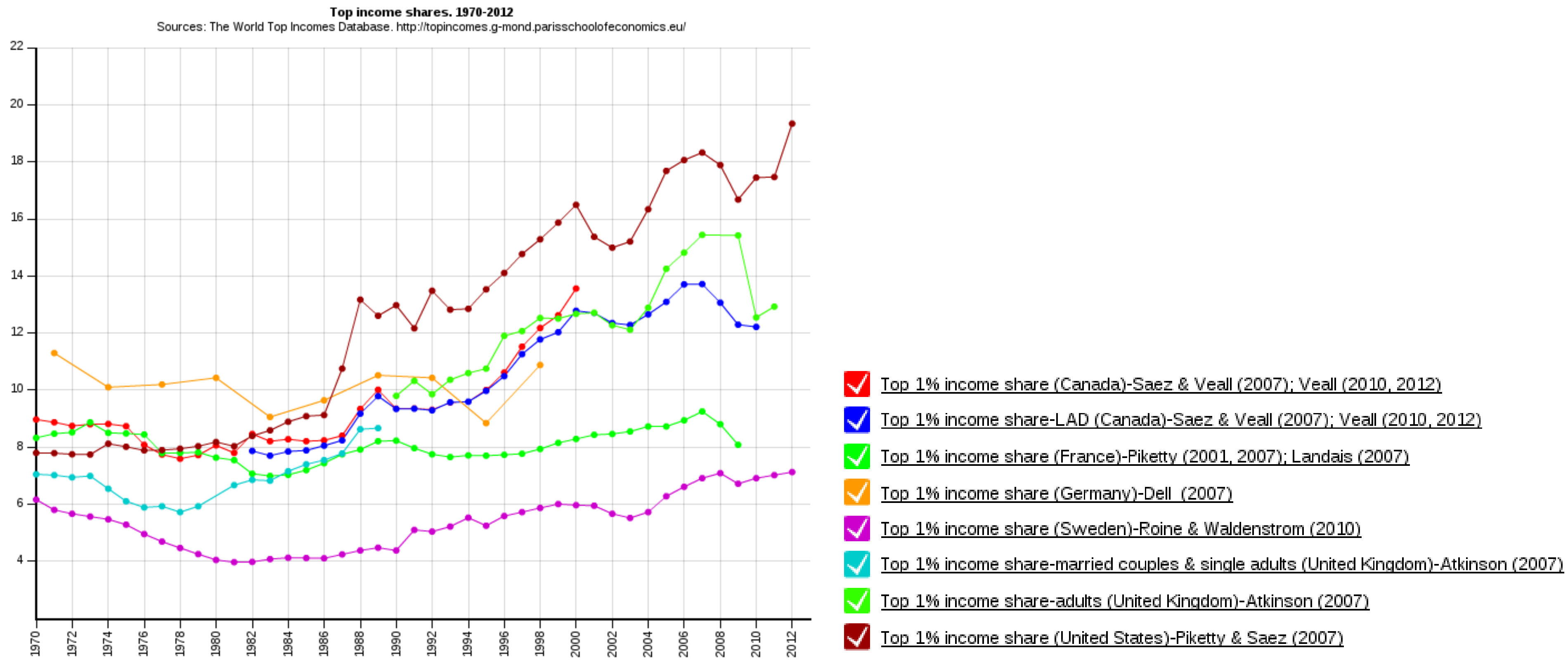
Piketty Capital in the Twenty-First Century, 2014, wealth, income, wage



The rise of income inequality since the 1970s is largely due to the rise of wage inequality.
Sources and series: see piketty.pse.ens.fr/capital21c.

Top Income Shares

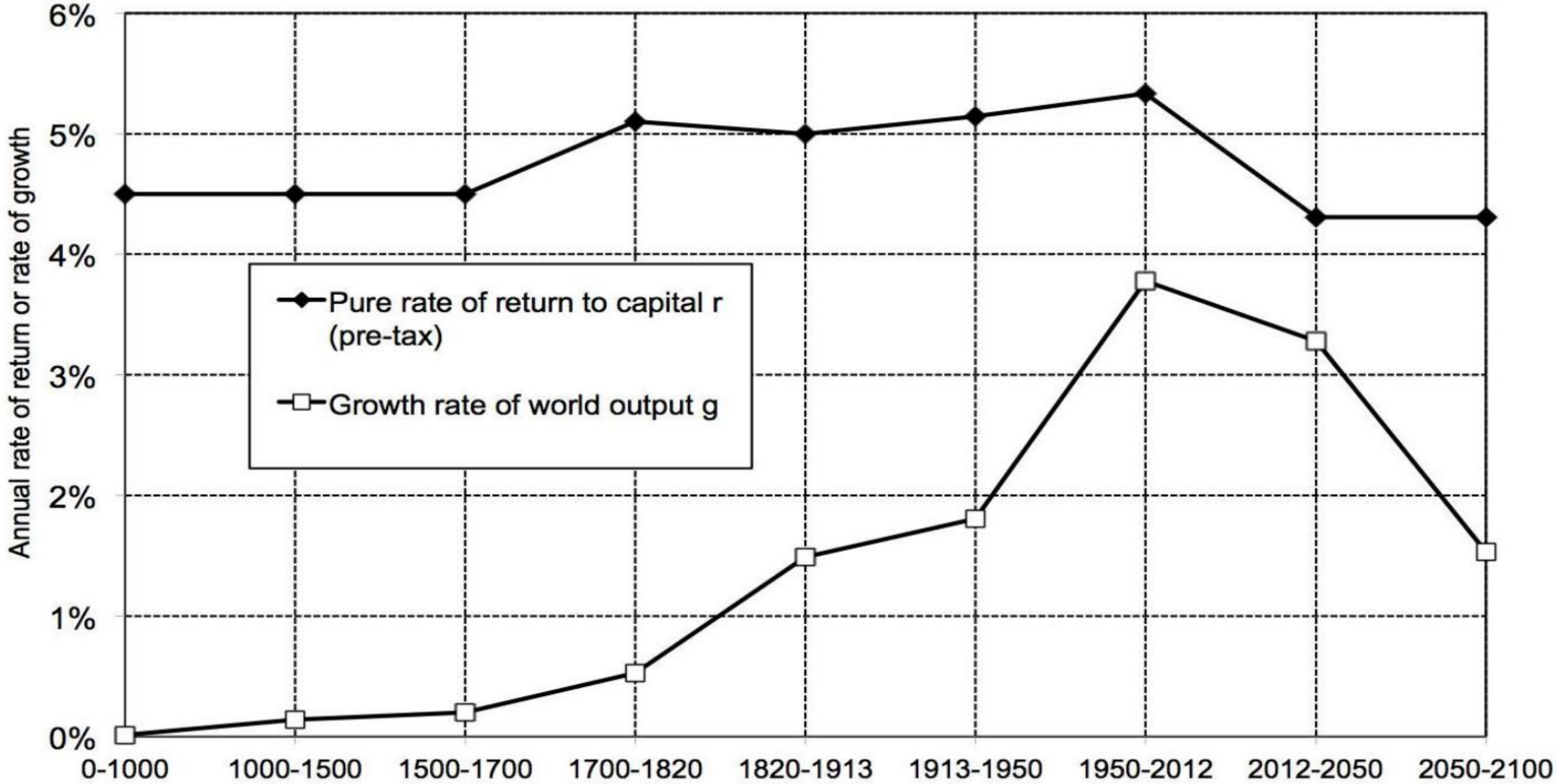
Piketty Capital in the Twenty-First Century, 2014



Fundamental Force of Divergence, $r > g$

Piketty Capital in the Twenty-First Century, 2014

Figure 10.9. Rate of return vs. growth rate at the world level, from Antiquity until 2100



The rate of return to capital (pre-tax) has always been higher than the world growth rate, but the gap was reduced during the 20th century, and might widen again in the 21st century.
Sources and series: see piketty.pse.ens.fr/capital21c

Poverty, in France

See Guélaud, Le nombre de pauvres a augmenté de 440.000 en France en 2010, 2012

La dernière enquête de l'Insee sur les niveaux de vie, rendue publique vendredi 7 septembre, est explosive. Que constate-t-elle en effet ? Qu'en 2010, le niveau de vie médian (19 270 euros annuels) a diminué de 0,5% par rapport à 2009, que seuls les plus riches s'en sont sortis et que la pauvreté, en hausse, frappe désormais 8,6 millions de personnes, soit 440 000 de plus qu'un an plus tôt.

Avec la fin du plan de relance, les effets de la crise se sont fait sentir massivement. En 2009, la récession n'avait que ralenti la progression en euros constants du niveau de vie médian (+ 0,4%, contre + 1,7% par an en moyenne de 2004 à 2008). Il faut remonter à 2004, précise l'Insee, pour trouver un recul semblable à celui de 2010 (0,5%).

Poverty, in France

La timide reprise économique de 2010 n'a pas eu d'effets miracle, puisque pratiquement toutes les catégories de la population, y compris les classes moyennes ou moyennes supérieures, ont vu leur niveau de vie baisser. N'a augmenté que celui des 5% des Français les plus aisés.

Dans un pays qui a la passion de l'égalité, la plupart des indicateurs d'inégalités sont à la hausse. L'indice de Gini, qui mesure le degré d'inégalité d'une distribution (en l'espèce, celle des niveaux de vie), a augmenté de 0,290 à 0,299 (0 correspondant à l'égalité parfaite et 1 à l'inégalité la plus forte). Le rapport entre la masse des niveaux de vie détenue par les 20 % les plus riches et celle détenue par les 20 % les plus modestes est passé de 4,3 à 4,5.

Poverty, in France

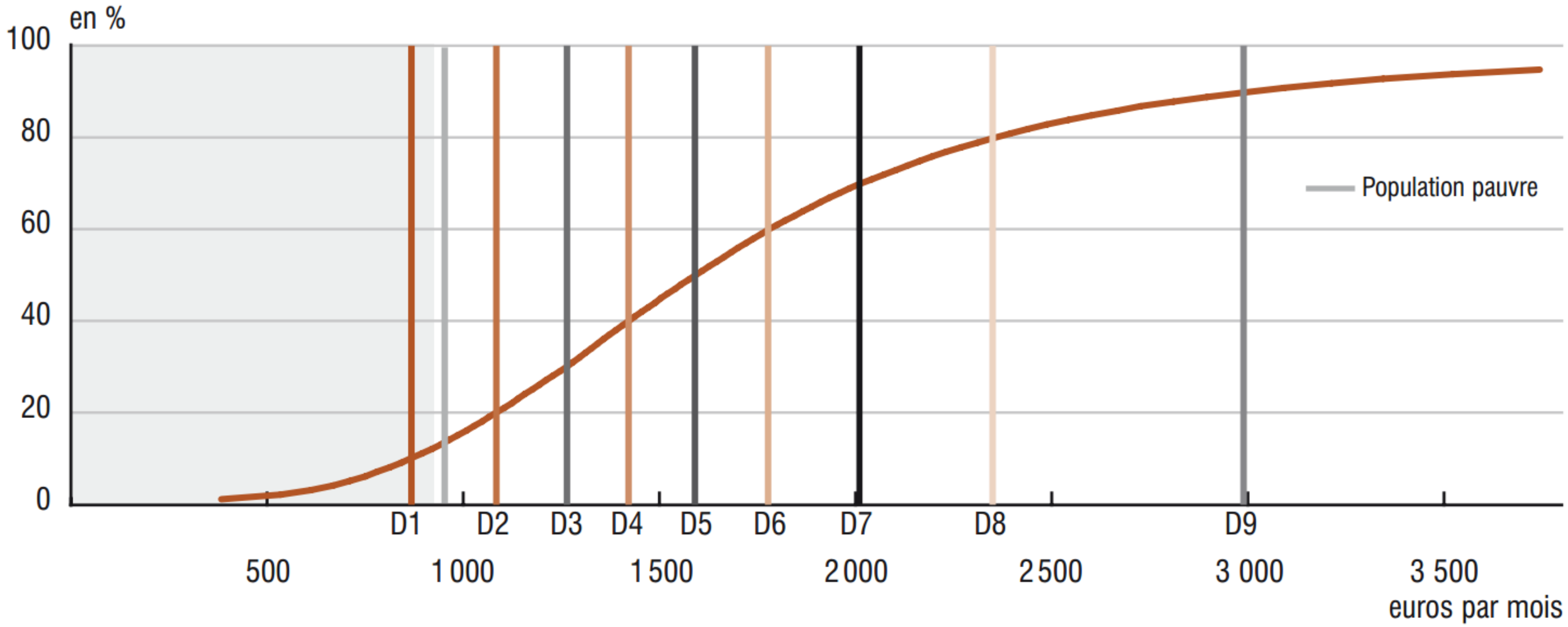
Déjà en hausse de 0,5 point en 2009, le taux de pauvreté monétaire a augmenté en 2010 de 0,6 point pour atteindre 14,1%, soit son plus haut niveau depuis 1997. 8,6 millions de personnes vivaient en 2010 en-dessous du seuil de pauvreté monétaire (964 euros par mois). Elles n'étaient que 8,1 millions en 2009. Mais il y a pire : une personne pauvre sur deux vit avec moins de 781 euros par mois

En 2010, le chômage a peu contribuéà l'augmentation de la pauvreté (les chômeurs représentent à peine 4% de l'accroissement du nombre des personnes pauvres). C'est du coté des inactifs qu'il faut plutôt se tourner : les retraités (11%), les adultes inactifs autres que les étudiants et les retraites (16%) - souvent les titulaires de minima sociaux - et les enfants. Les moins de 18 ans contribuent pour près des deux tiers (63%) à l'augmentation du nombre de personnes pauvres [...]

Incomes in France

See Houdré, Missègue & Seguin Inégalités de niveau de vie et pauvreté, 2012

1. Répartition de la population selon le niveau de vie en 2009

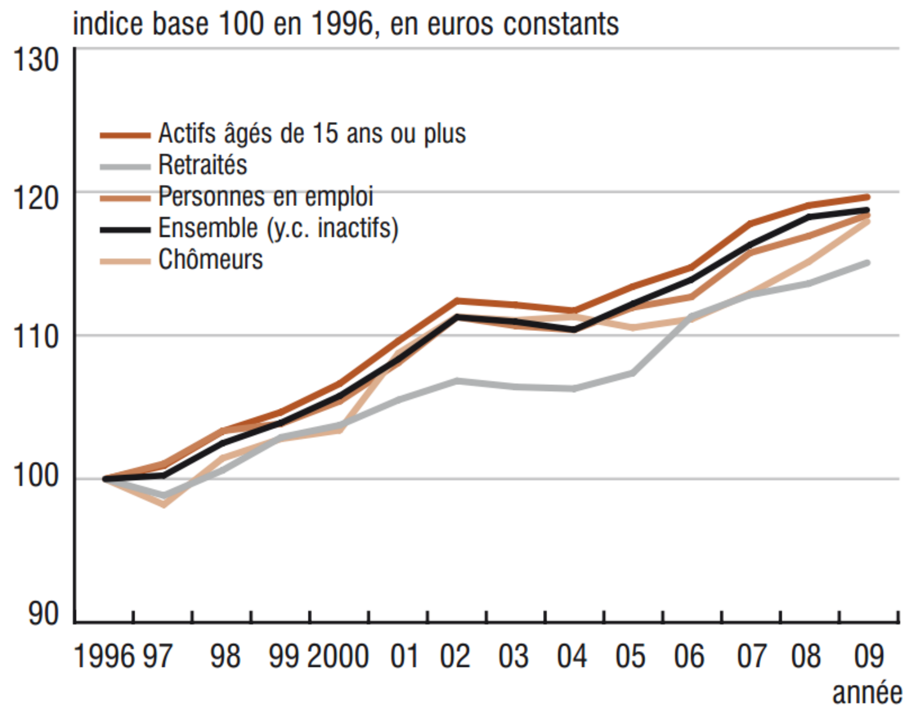


Champ : France métropolitaine, personnes vivant dans un ménage dont le revenu déclaré au fisc est positif ou nul et dont la personne de référence n'est pas étudiante.
Lecture : D1 à D9 désignent les 9 déciles de niveaux de vie, seuils qui partagent la population en 10 sous-populations d'effectifs égaux. 70 % des personnes vivent avec moins de 2 010 euros par mois.
Sources : Insee ; DGFiP ; Cnaf ; Cnav ; CCMSA, enquête Revenus fiscaux et sociaux 2009.

Incomes in France

See Houdré, Missègue & Seguin Inégalités de niveau de vie et pauvreté, 2012

2. Évolution du niveau de vie médian selon la situation sur le marché du travail

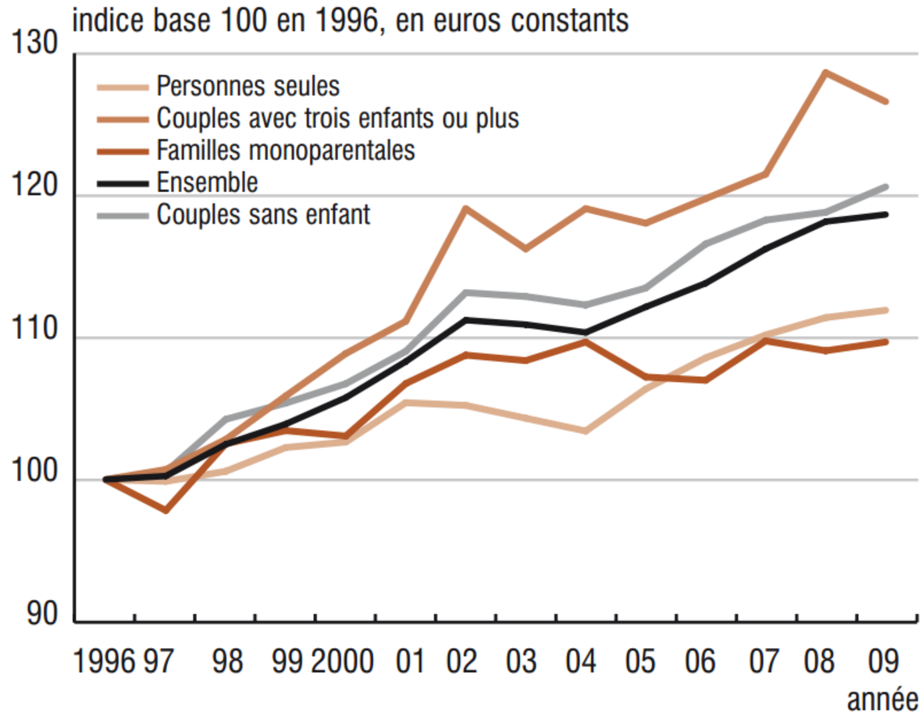


Champ : France métropolitaine, personnes vivant dans un ménage dont le revenu déclaré au fisc est positif ou nul et dont la personne de référence n'est pas étudiante.
Lecture : de 1996 à 2009, le niveau de vie médian des personnes ayant un emploi augmente de 20 %, soit une progression de 1,4 % en moyenne par an.
Sources : Insee ; DGI, enquêtes Revenus fiscaux et sociaux rétropolées 1996 à 2004 - Insee ; DGFiP ; Cnaf ; Cnav ; CCMSA, enquêtes Revenus fiscaux et sociaux 2005 à 2009.

Incomes in France

See Houdré, Missègue & Seguin Inégalités de niveau de vie et pauvreté, 2012

3. Évolution du niveau de vie médian selon quelques configurations familiales



Champ : France métropolitaine, personnes vivant dans un ménage dont le revenu déclaré au fisc est positif ou nul et dont la personne de référence n'est pas étudiante.
Lecture : de 1996 à 2009, le niveau de vie médian des personnes vivant en couple avec 3 enfants ou plus augmente de 26,7 %, soit une progression de 1,8 % en moyenne par an.
Sources : Insee ; DGI, enquêtes Revenus fiscaux et sociaux rétropolées 1996 à 2004 - Insee ; DGFiP ; Cnaf ; Cnav ; CCMSA, enquêtes Revenus fiscaux et sociaux 2005 à 2009.

Incomes in France

See Houdré, Missègue & Seguin Inégalités de niveau de vie et pauvreté, 2012

4. Contribution à l'augmentation de la pauvreté selon l'activité des personnes

	2008			2009			Contribution à l'augmentation du nombre de personnes pauvres
	Ensemble des personnes	Pauvreté au seuil de 60 % de la médiane		Ensemble des personnes	Pauvreté au seuil de 60 % de la médiane		
		Personnes pauvres	Taux de pauvreté		Personnes pauvres	Taux de pauvreté	
	(milliers)	(milliers)	(%)	(milliers)	(milliers)	(%)	(%)
Actifs de 18 ans ou plus	**27 687**	**2 634**	**9,5**	**27 726**	**2 796**	**10,1**	**48**
Personnes en emploi	25 530	1 863	7,3	25 050	1 866	7,4	1
Chômeurs	2 156	772	35,8	2 677	930	34,7	47
Inactifs de 18 ans ou plus	**19 063**	**2 873**	**15,1**	**19 278**	**2 990**	**15,5**	**34**
Étudiants	1 789	324	18,1	1 726	351	20,3	8
Retraités	12 960	1 283	9,9	13 163	1 308	9,9	7
Autres inactifs	4 315	1 266	29,3	4 389	1 331	30,3	19
Enfants de moins de 18 ans	**13 436**	**2 328**	**17,3**	**13 475**	**2 387**	**17,7**	**18**
Ensemble de la population	**60 186**	**7 836**	**13**	**60 479**	**8 173**	**13,5**	**100**

Champ : France métropolitaine, personnes vivant dans un ménage dont le revenu déclaré au fisc est positif ou nul et dont la personne de référence n'est pas étudiante.
Lecture : entre 2008 et 2009, le nombre d'actifs de 18 ans et plus en situation de pauvreté passe de 2,634 millions de personnes à 2,796 millions de personnes. Cette évolution contribue à hauteur de 48 % à l'augmentation totale de la population pauvre.
Sources : Insee ; DGFiP ; Cnaf ; Cnav ; CCMSA, enquêtes Revenus fiscaux et sociaux 2008 et 2009.

Incomes in France

See Houdré, Missègue & Seguin Inégalités de niveau de vie et pauvreté, 2012

9. Décomposition du revenu disponible du ménage selon son groupe de niveau de vie

en %

Type de revenus perçus[1]	Groupes de niveaux de vie						Ensemble
	Modestes	Intermédiaires			Aisés		
	1	2	3	4	5	6	
Salaires et allocations chômage	43,1	60,3	68,7	66,3	60,7	48,3	**61,2**
Revenus d'activité indépendants	2,3	2,2	2,9	5,7	12,5	18,4	**5,5**
Pensions et retraites	26,8	28,3	23,7	23,2	18,8	11,0	**23,6**
Revenus du patrimoine	3,2	4,4	6,4	11,3	20,0	44,6	**10,8**
Transferts et prélèvements[2]	24,5	4,8	–1,7	–6,6	–12,1	–22,4	**–1,0**

1. Nets de CSG et de CRDS.
2. Prestations sociales nettes de CRDS (allocations familiales, allocations logement, minima sociaux) auxquelles sont retranchés les impôts directs suivants : impôts sur le revenu, taxe d'habitation, prélèvements libératoires sur valeurs mobilières.
Champ : France métropolitaine, personnes vivant dans un ménage dont le revenu déclaré au fisc est positif ou nul et dont la personne de référence n'est pas étudiante.
Lecture : Les revenus du patrimoine représentent 6,4 % du revenu disponible des ménages du groupe 3.
Sources : Insee ; DGFiP ; Cnaf ; Cnav ; CCMSA, enquête Revenus fiscaux et sociaux 2009.

Incomes in France

See Houdré, Missègue & Seguin Inégalités de niveau de vie et pauvreté, 2012

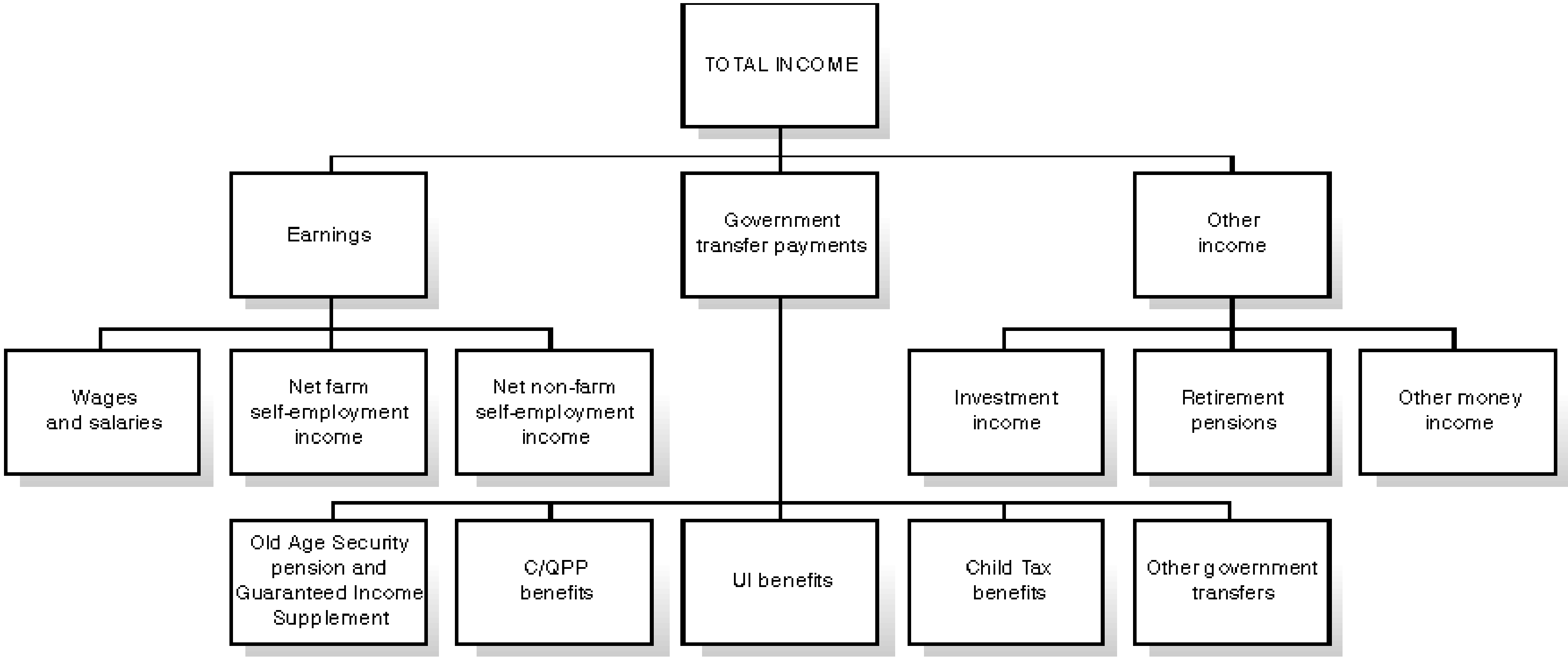
2. Niveau de vie annuel et indicateurs d'inégalité de 1996 à 2009

	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Niveau de vie en euros 2009														
D5 (médian)	16 070	16 110	16 470	16 700	17 000	17 410	17 880	17 830	17 740	18 030	18 300	18 690	19 000	19 080
Indicateurs d'inégalité														
D9/D1	3,5	3,5	3,4	3,4	3,5	3,4	3,4	3,4	3,3	3,3	3,4	3,4	3,4	3,4
D9/D5	1,9	1,9	1,9	1,9	1,9	1,9	1,9	1,9	1,9	1,9	1,9	1,9	1,9	1,9
D5/D1	1,9	1,9	1,8	1,8	1,8	1,8	1,8	1,8	1,8	1,8	1,8	1,8	1,8	1,8
S20 (%)	9,0	9,0	9,2	9,1	9,1	9,1	9,3	9,3	9,3	9,1	9,0	9,0	9,0	8,9
S50 (%)	31,1	31,0	31,2	30,9	30,8	30,8	31,1	31,2	31,2	31,0	30,7	30,7	30,9	30,7
S80 (%)	63,0	63,0	63,0	62,3	62,0	62,1	62,3	62,4	62,4	62,1	61,6	61,8	61,7	61,8
(100-S80)/S20	4,1	4,1	4,0	4,1	4,2	4,2	4,1	4,0	4,0	4,2	4,3	4,2	4,3	4,3
Indice de Gini	0,279	0,279	0,276	0,284	0,286	0,286	0,281	0,280	0,281	0,286	0,291	0,289	0,289	0,290

Champ : France métropolitaine, personnes vivant dans un ménage dont le revenu déclaré au fisc est positif ou nul et dont la personne de référence n'est pas étudiante.
Lecture : en 2009, la moitié des personnes disposent d'un niveau de vie annuel inférieur à 19 080 euros. Le rapport entre le niveau de vie plancher des 10 % des personnes les plus aisées et le niveau de vie plafond des 10 % les plus modestes s'élève à 3,4. Les 20 % les plus pauvres ont 8,9 % de la masse des niveaux de vie (S20). Les 20 % les plus aisées ont 38,2 % de la masse des niveaux de vie (1-S80).
Sources : Insee-DGI, enquêtes Revenus fiscaux et sociaux rétropolées 1996 à 2004 ; Insee-DGFiP-Cnaf-Cnav-CCMSA, enquêtes Revenus fiscaux et sociaux 2005 à 2009.

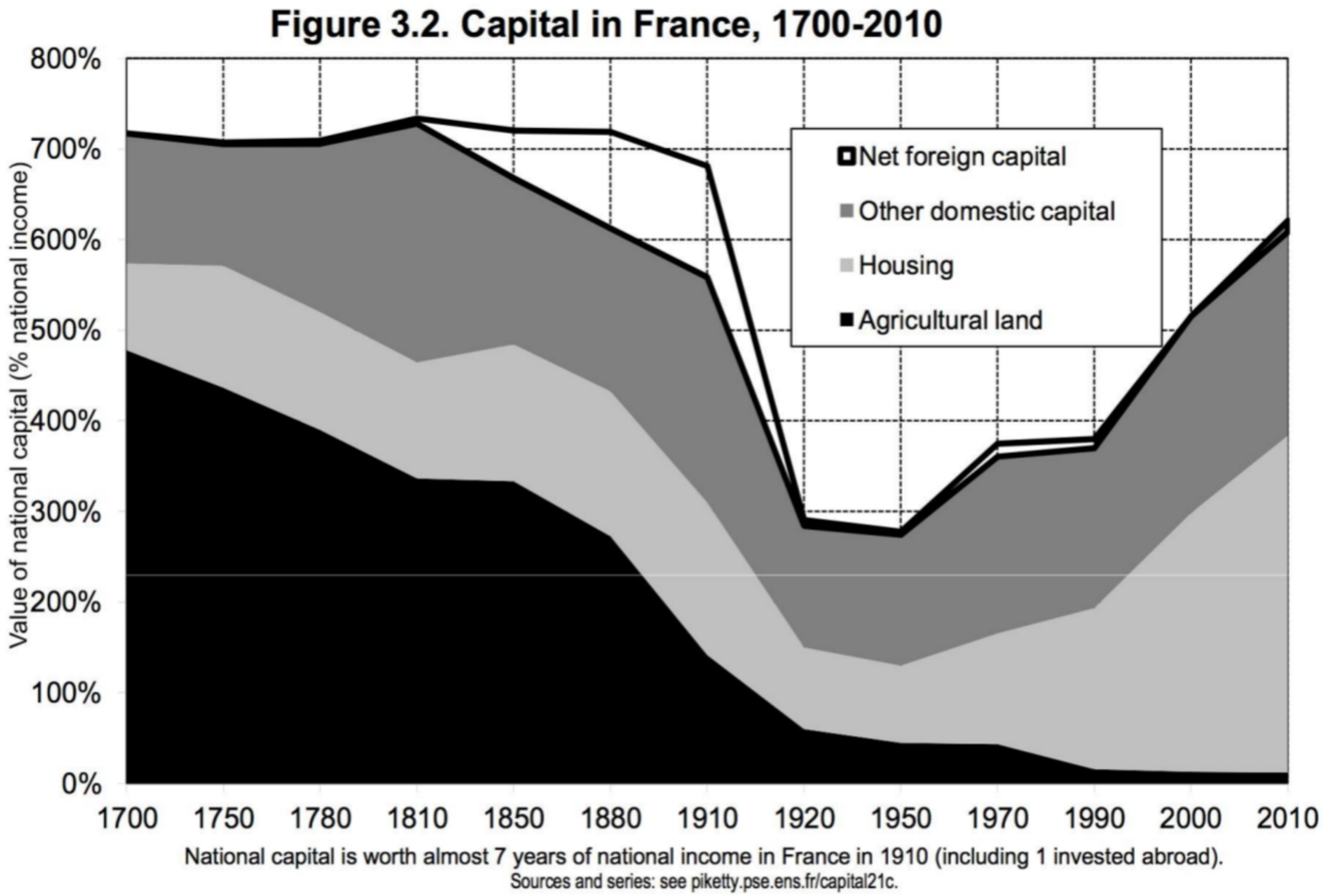
Income ?

See Statistics Canada Total Income, via Flachaire (2015).



Income ? Micro vs macro

Piketty Capital in the Twenty-First Century, 2014,



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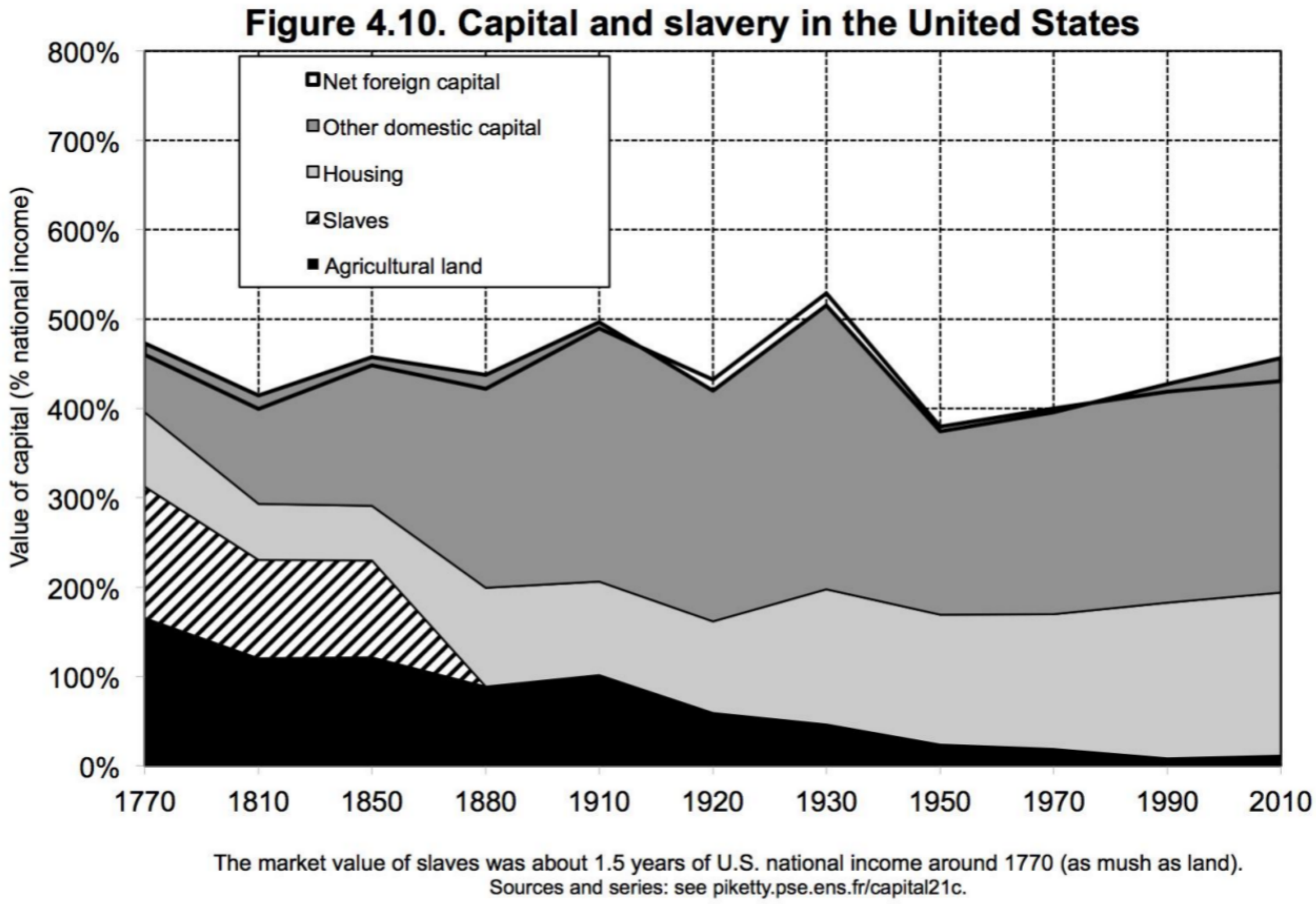


Income ? Micro vs macro

To compare various household incomes

- Oxford scale (OECD equivalent scale)
  - 1.0 to the first adult
  - 0.7 to each additional adult (aged 14, and more)
  - 0.5 to each child
- OECD-modified equivalent scale (late 90s by eurostat)
  - 1.0 to the first adult
  - 0.5 to each additional adult (aged 14, and more)
  - 0.3 to each child
- More recent OECD scale
  - square root of household size

Income ? Micro vs macro

Household	OECD equivalent scale	OECD-modified scale	Square root scale
	$\frac{\text{income}}{1+0.7+3*0.5}$	$\frac{\text{income}}{1+0.5+3*0.3}$	$\frac{\text{income}}{\sqrt{5}}$
	$\frac{\text{income}}{1+0.7}$	$\frac{\text{income}}{1+0.5}$	$\frac{\text{income}}{\sqrt{2}}$
	$\frac{\text{income}}{1+0.7+0.5}$	$\frac{\text{income}}{1+0.5+0.3}$	$\frac{\text{income}}{\sqrt{3}}$
	$\frac{\text{income}}{1+0.7+2*0.5}$	$\frac{\text{income}}{1+0.5+2*0.3}$	$\frac{\text{income}}{\sqrt{4}}$

Income ? Tax Issues

E.g. total taxes paid by total wage

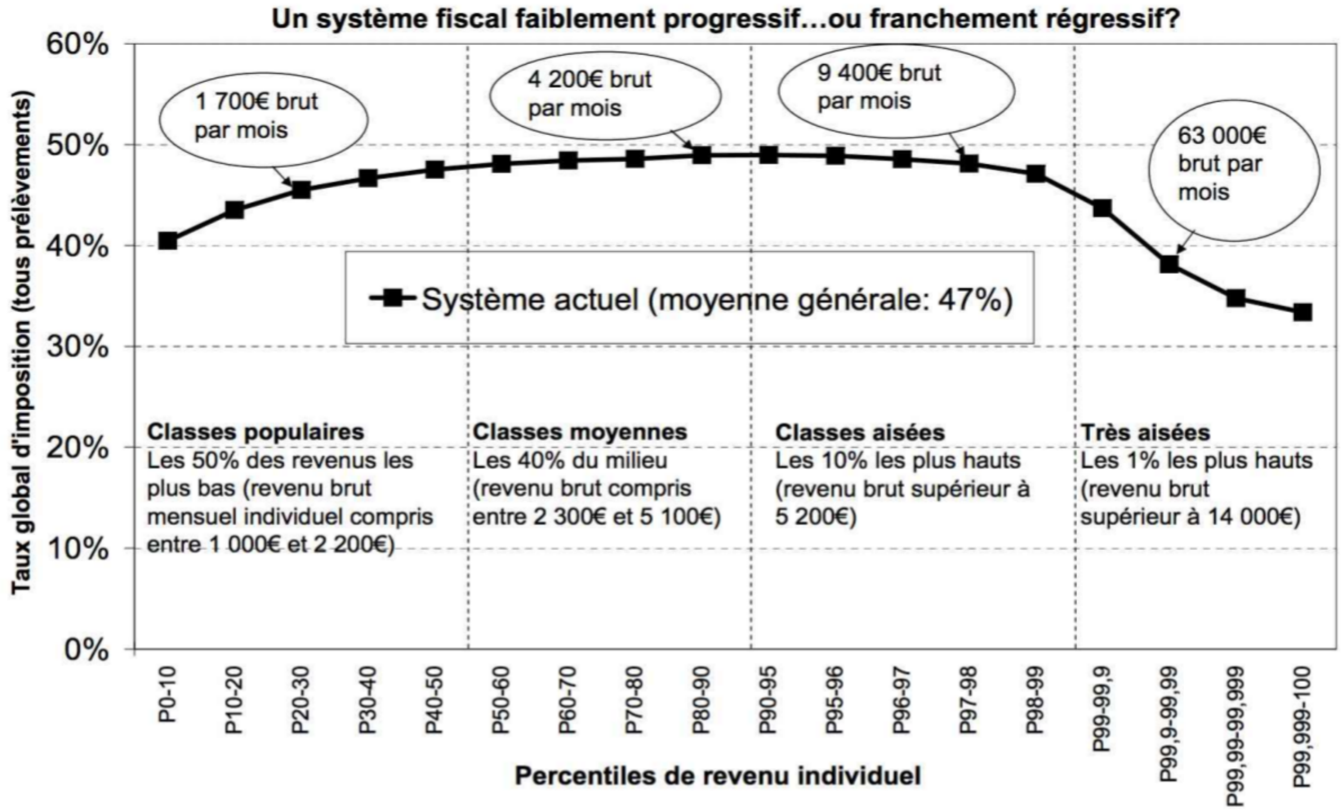
Tax burdens around the world

Country	Single, no kids	Married, 2 kids	Country	Single, no kids	Married, 2 kids
Australia	28.3%	16.0%	Korea	17.3%	16.2%
Austria	47.4%	35.5%	Luxembourg	35.3%	12.2%
Belgium	55.4%	40.3%	Mexico	18.2%	18.2%
Canada	31.6%	21.5%	Netherlands	38.6%	29.1%
Czech Republic	43.8%	27.1%	New Zealand	20.5%	14.5%
Denmark	41.4%	29.6%	Norway	37.3%	29.6%
Finland	44.6%	38.4%	Poland	43.6%	42.1%
France	50.1%	41.7%	Portugal	36.2%	26.6%
Germany	51.8%	35.7%	Slovak Republic	38.3%	23.2%
Greece	38.8%	39.2%	Spain	39.0%	33.4%
Hungary	50.5%	39.9%	Sweden	47.9%	42.4%
Iceland	29.0%	11.0%	Switzerland	29.5%	18.6%
Ireland	25.7%	8.1%	Turkey	42.7%	42.7%
Italy	45.4%	35.2%	United Kingdom	33.5%	27.1%
Japan	27.7%	24.9%	United States	29.1%	11.9%

Source: OECD, 2005 data

Income ? Tax Issues

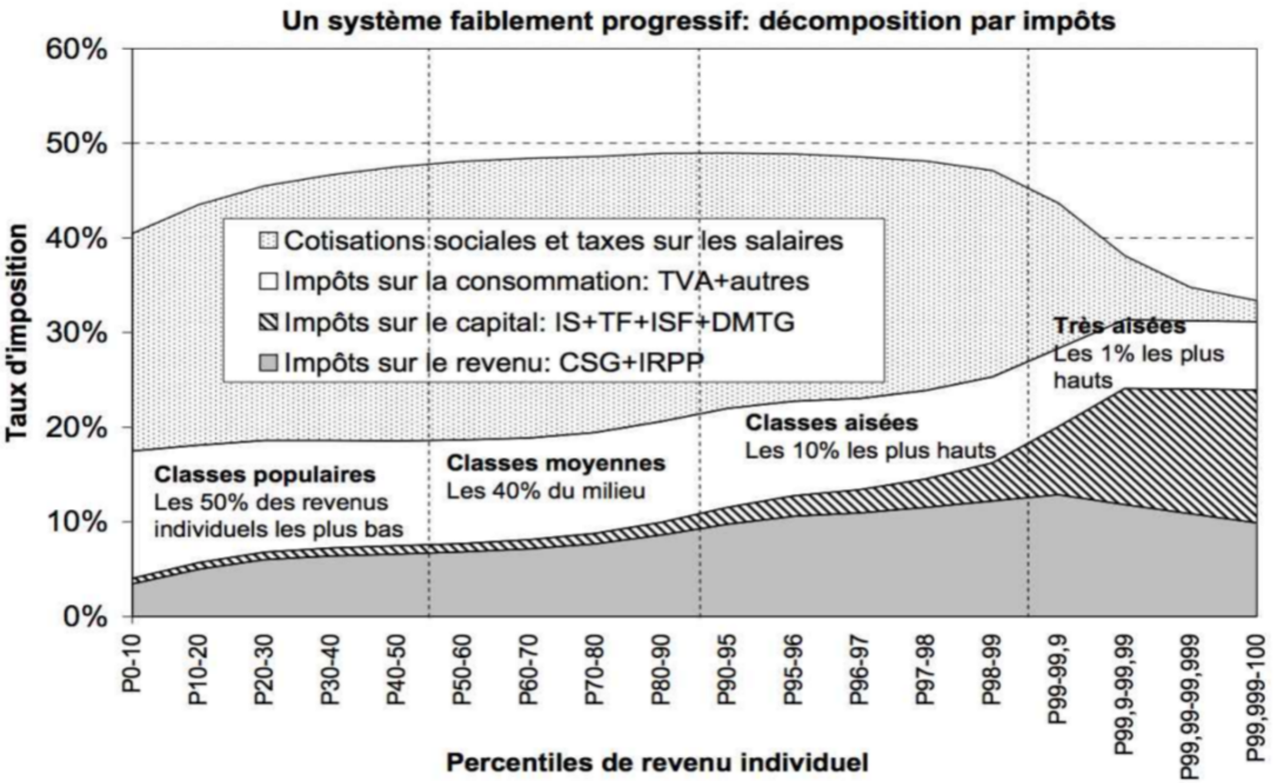
via Landais, Piketty & Saez Pour une révolution fiscale, 2011



Lecture: le graphique montre le taux global d'imposition (incluant tous les prélèvements) par groupe de revenus au sein de la population 18-65 ans travaillant à au moins 80% du plein temps. P0-10 désigne les percentiles 0 à 10, càd les 10% des personnes avec les revenus les plus faibles, P10-20 les 10% suivants, ..., P99.999-100 désigne les .001% les plus riches. La moyenne générale d'imposition est de 47% en moyenne. Les taux d'imposition croissent légèrement avec le revenu jusqu'au 95e percentile puis baissent avec le revenu pour les 5% les plus riches.
Source: C. Landais, T. Piketty & E. Saez, Pour une révolution fiscale, chapitre 1, p.50
Voir www.revolution-fiscale.fr, annexe au chapitre 1 (où nous montrons aussi les chiffres pour la population adulte totale).

Income ? Tax Issues

via Landais, Piketty & Saez Pour une révolution fiscale, 2011

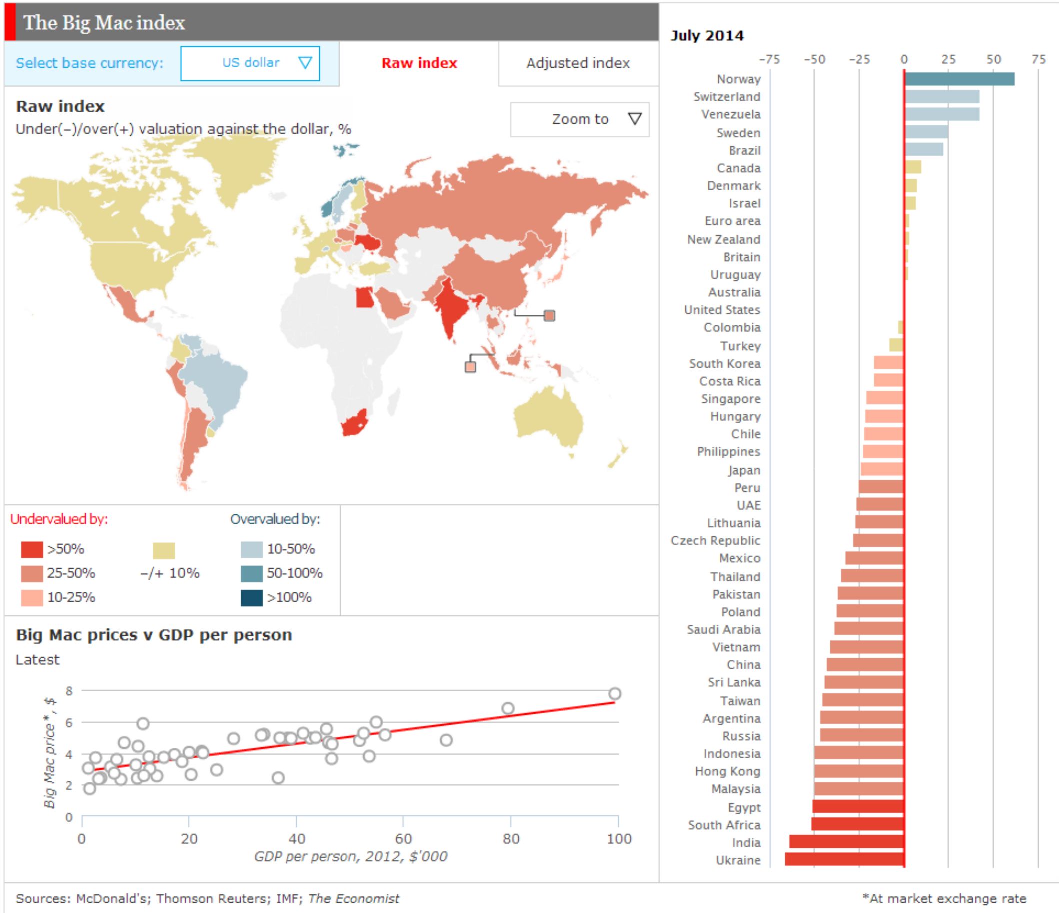


Lecture: le graphique montre le taux global d'imposition (incluant tous les prélèvements comme dans le graphique précédant) et sa décomposition par groupe de revenus au sein de la population 18-65 ans travaillant à au moins 80% du plein temps. Groupes de revenus: P0-10 désigne les percentiles 0 à 10, càd les 10% des personnes avec les revenus les plus faibles, P10-20 les 10% suivants, ..., P99.999-100 désigne les .001% les plus riches.

Le graphique décompose les impôts en quatre grandes catégories : cotisations sociales (et autres taxes sur les salaires), les impôts sur la consommation (TVA et autres impôts indirects), les impôts sur le capital (impôt sur les bénéfices des sociétés (IS), taxe foncière (TF), impôt sur la fortune (ISF) et droits de successions (DMTG)), et les impôts sur le revenu (CSG et IRPP).
Source: C. Landais, T. Piketty & E. Saez, Pour une révolution fiscale, chapitre 1, p.51
Source: Voir www.revolution-fiscale.fr, annexe au chapitre 1 (où nous montrons aussi les chiffres pour la population adulte totale).

International Comparisons, Puchasing Power Parity

See The Economist The Big Mac index, 2014



International Comparisons, Puchasing Power Parity

See The Economist The Big Mac index, 2014, via Flachaire

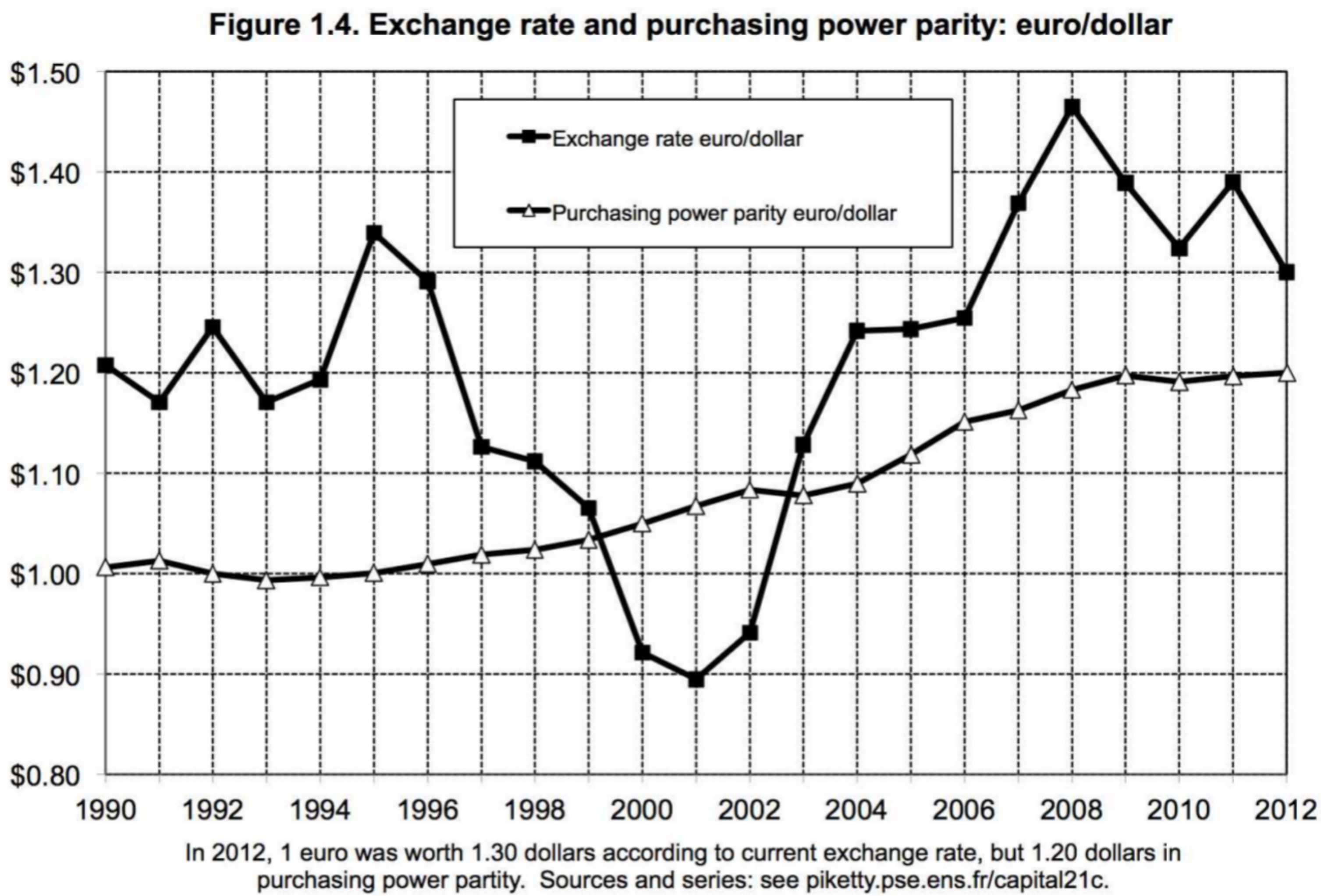
	Price in local currency	exchange rate: Actual dollar exchange rate	exchange rate: Price in dollars†	PPP: Implied PPP of the dollars‡	PPP: Price in dollars	Under(-)/over(+) valuation against the dollar, %
United States	4.80	1.00	4.80	1.00	4.80	
Australia	5.10	1.06	4.81	1.06	4.80	0.40
Britain	2.89	0.59	4.93	0.60	4.80	2.71
Canada	5.64	1.07	5.25	1.18	4.80	9.51
China	16.90	6.20	2.73	3.52	4.80	-43.14
India	105	60.09	1.75	21.90	4.80	-63.56
Norway	48	6.19	7.76	10.01	4.80	61.79
South Africa	24.50	10.51	2.33	5.11	4.80	-51.41
Sweden	40.7	6.84	5.95	8.49	4.80	24.17
Switzerland	6.16	0.90	6.83	1.28	4.80	42.36
Ukraine	19	11.69	1.63	3.96	4.80	-66.09
France	3.90	0.74	5.25	0.81	4.80	9.51
Germany	3.67	0.74	4.94	0.77	4.80	3.05
Spain	3.65	0.74	4.91	0.76	4.80	2.49

† Big Mac price in \$ = Big Mac price in local currency / actual \$ exchange rate

‡ Implied PPP of the \$ = Big Mac price in local currency / Big Mac price in the U.S. (4.80)

International Comparisons, Puchasing Power Parity

Piketty Capital in the Twenty-First Century, 2014, wealth, income, wage



From Income and Wealth to Human Development

The Human Development Index (HDI, see wikipedia) is a composite statistic of life expectancy, education, and income indices used to rank countries into four tiers of human development. It was created by Indian economist Amartya Sen and Pakistani economist Mahbub ul Haq in 1990, and was published by the United Nations Development Programme.

The HDI is a composite index at value between 0 (awful) and 1 (perfect) based on the mixing of three basic indices aiming at representing on an equal footing measures of helth, education and standard of living.

HDI Computation, new method (2010)

Published on 4 November 2010 (and updated on 10 June 2011), starting with the 2010 Human Development Report the HDI combines three dimensions :

- A long and healthy life : Life expectancy at birth
- An education index : Mean years of schooling and Expected years of schooling
- A decent standard of living : GNI per capita (PPP US$)

In its 2010 Human Development Report, the UNDP began using a new method of calculating the HDI. The following three indices are used.

The idea is to define a x index as

$$x_{\text{ index}} = \frac{x - \min(x)}{\max(x) - \min(x)}$$

1. Health, Life Expectancy Index (LEI) $= \dfrac{\text{LE} - 20}{85 - 20}$

where LE is Life Expectancy at birth

HDI Computation, new method (2010)

2. Education, Education Index (EI) $= \dfrac{\text{MYSI} + \text{EYSI}}{2}$

2.1 Mean Years of Schooling Index (MYSI) $= \dfrac{\text{MYS}}{15}$

where MYS is the Mean years of schooling (Years that a 25-year-old person or older has spent in schools)

2.2 Expected Years of Schooling Index (EYSI) $= \dfrac{\text{EYS}}{18}$

EYS : Expected years of schooling (Years that a 5-year-old child will spend with his education in his whole life)

3. Standard of Living Income Index (II) $= \dfrac{\log(\text{GNIpc}) - \log(100)}{\log(75,000) - \log(100)}$

where GNIpc : Gross national income at purchasing power parity per capita

Finally, the HDI is the geometric mean of the previous three normalized indices : HDI $= \sqrt[3]{\text{LEI} \cdot \text{EI} \cdot \text{II}}$.

Economic Well-Being

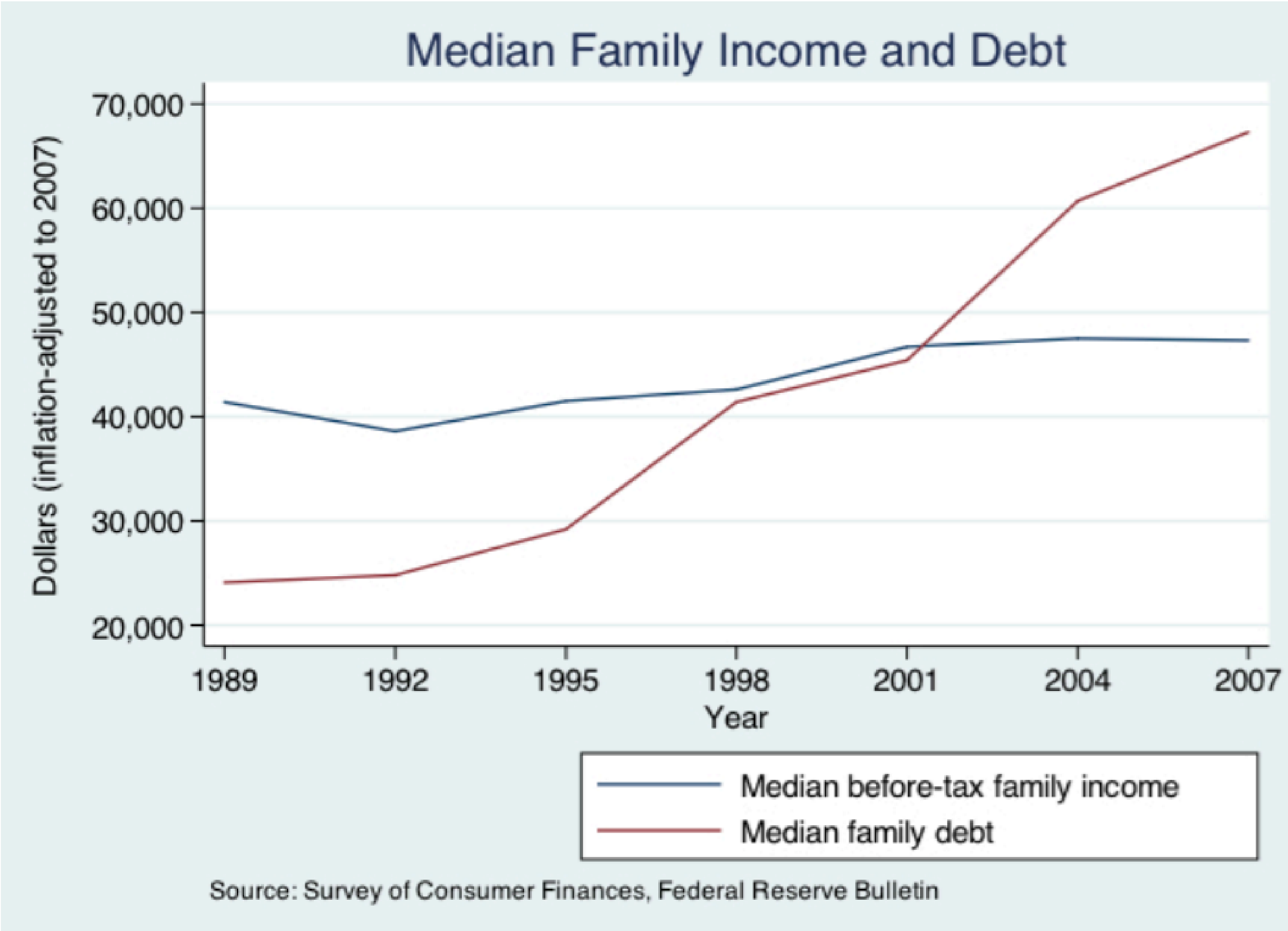
See Osberg The Measurement of Economic Well-Being, 1985 and

Osberg & Sharpe New Estimates of the Index of Economic Well-being, 2002

See also Jank & Owens Inequality in the United States, 2013, for stats and graphs about inequalities in the U.S., in terms of health, education, crime, etc.

Various Aspects of Inequalities in the U.S.

Jank & Owens Inequality in the United States, 2013



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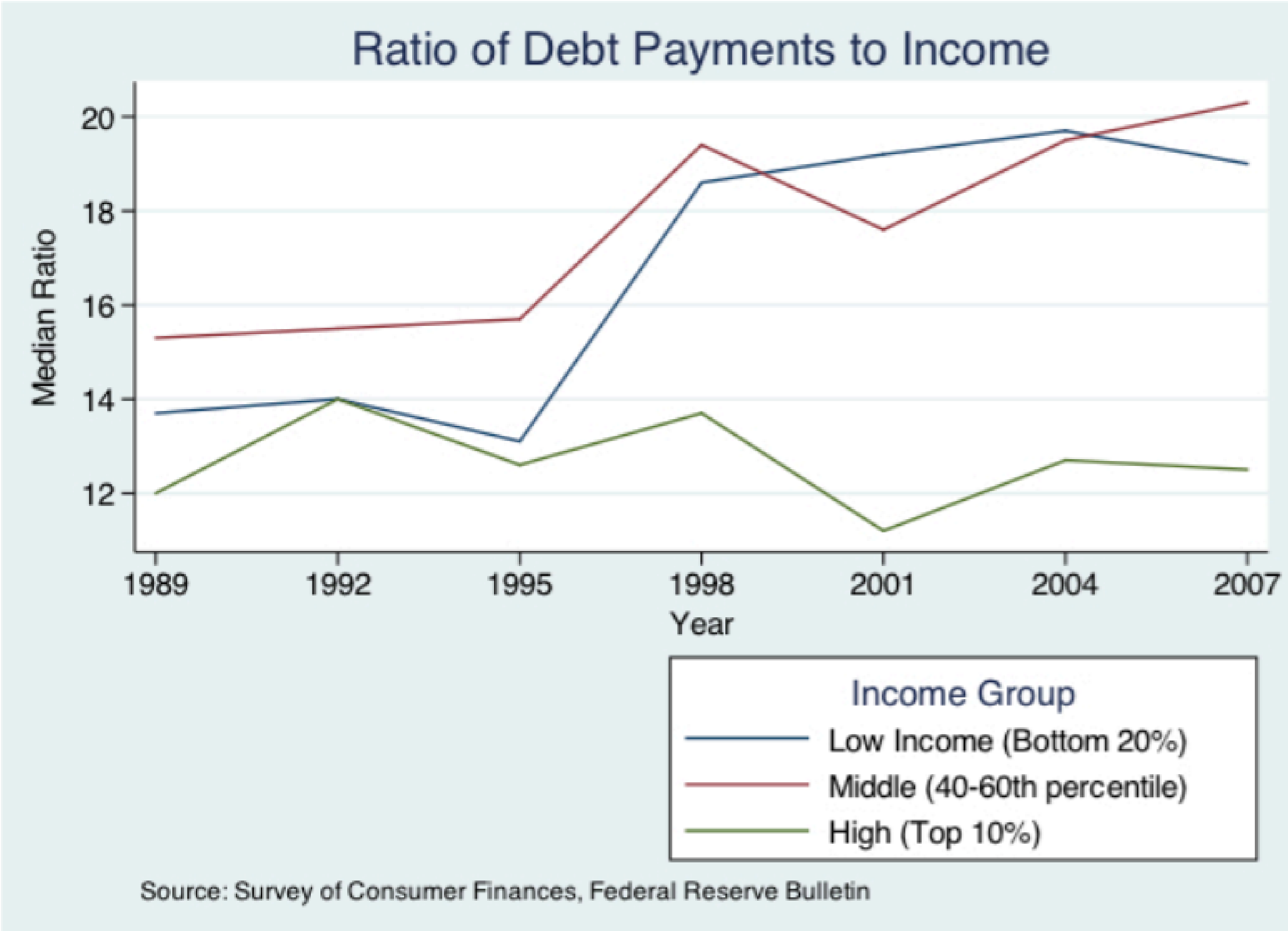

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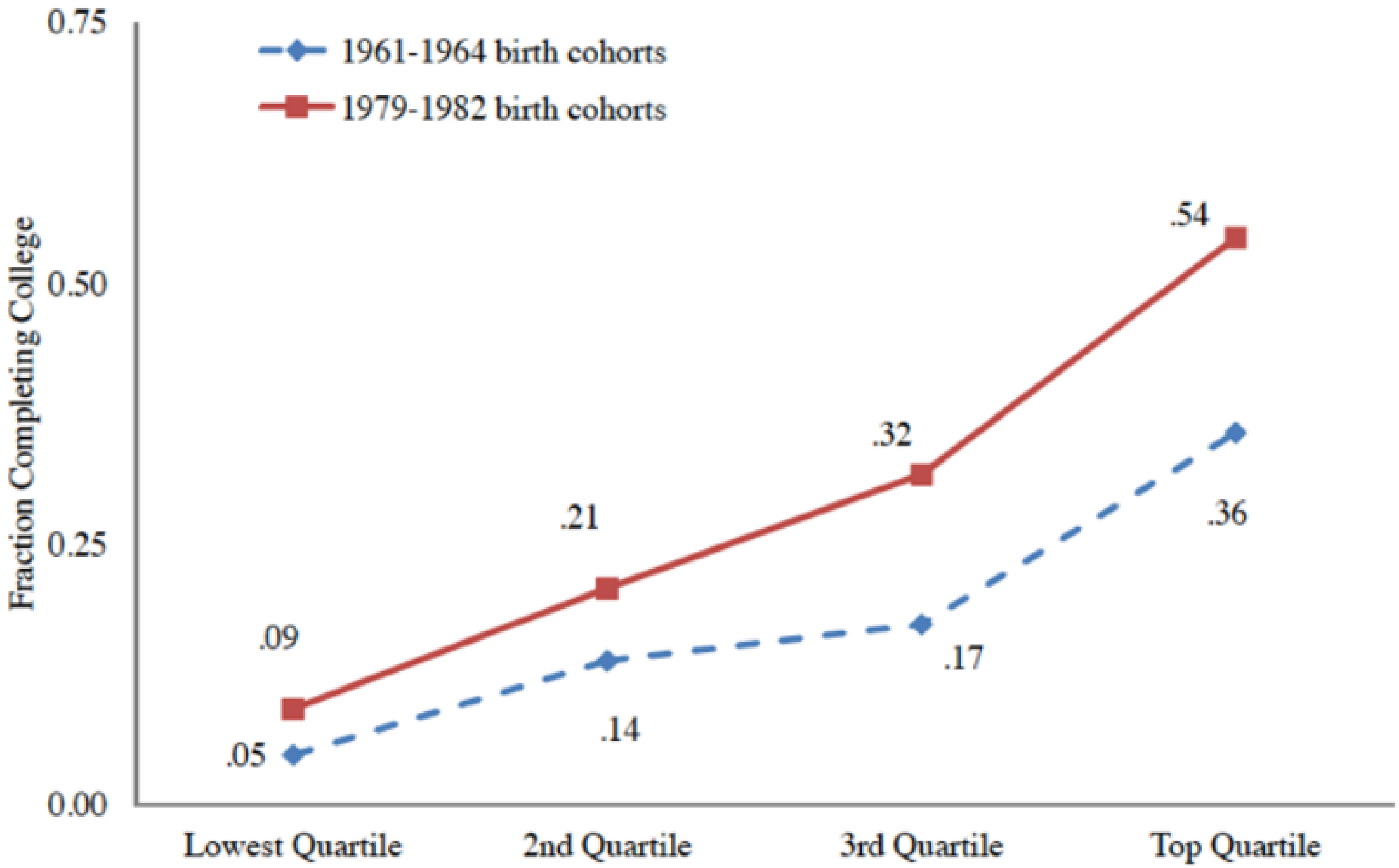

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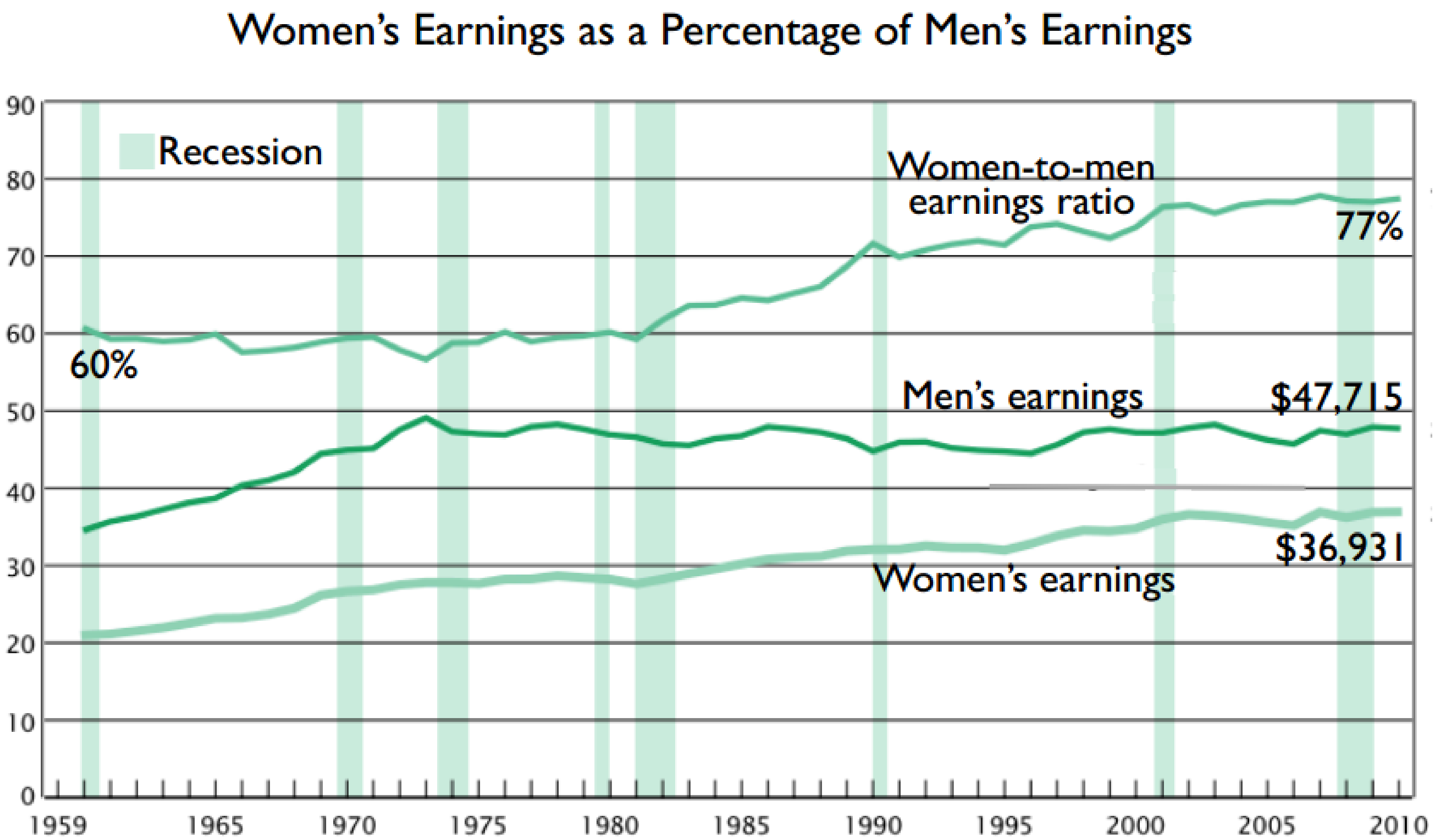

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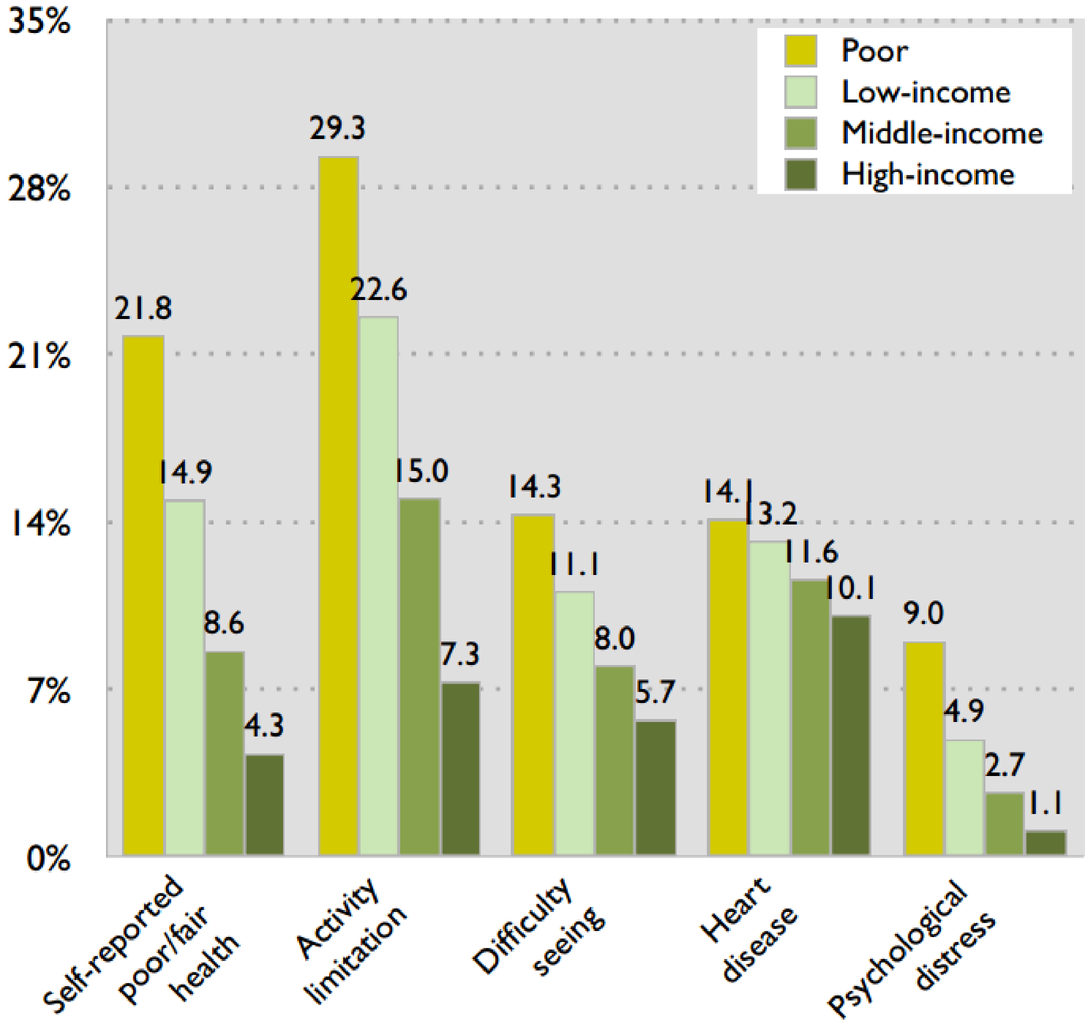

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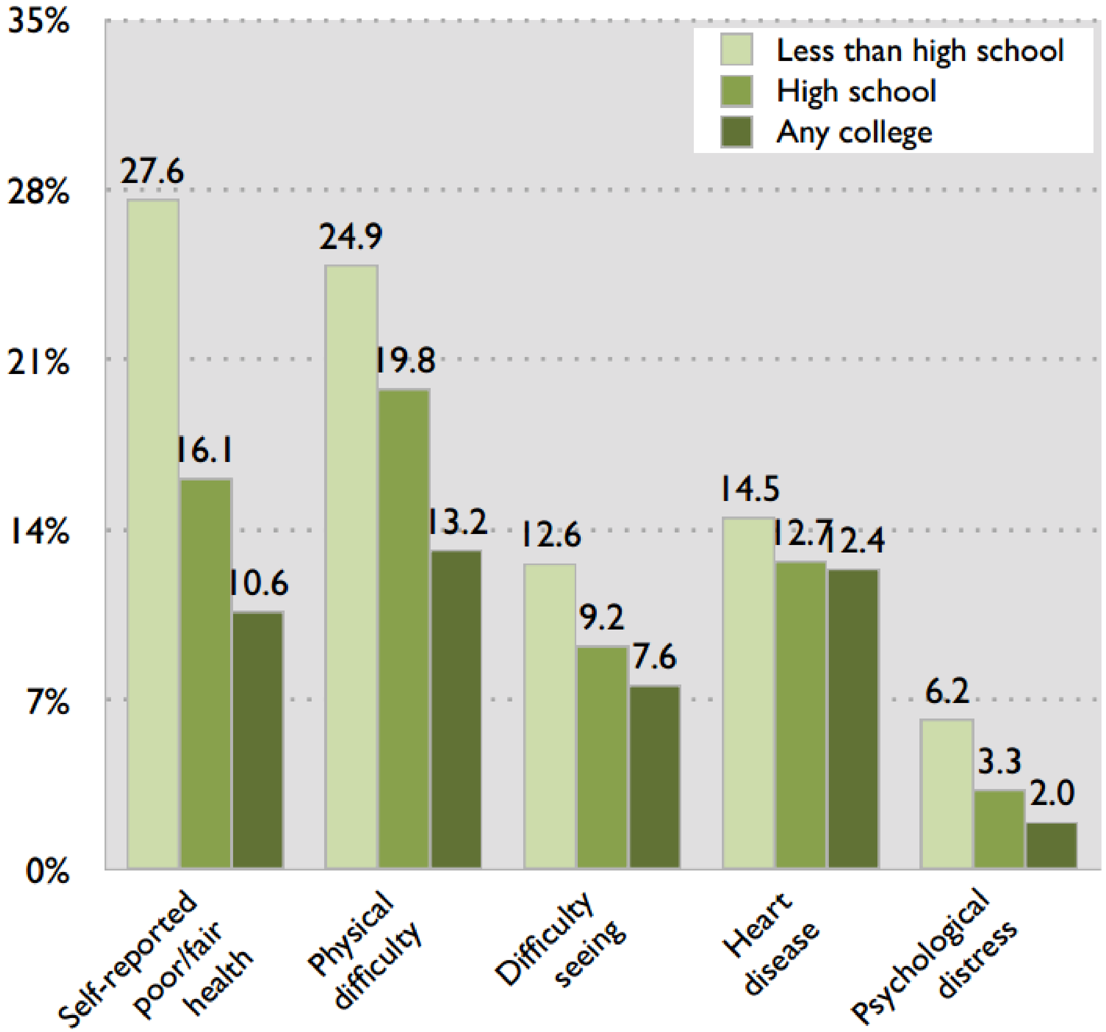

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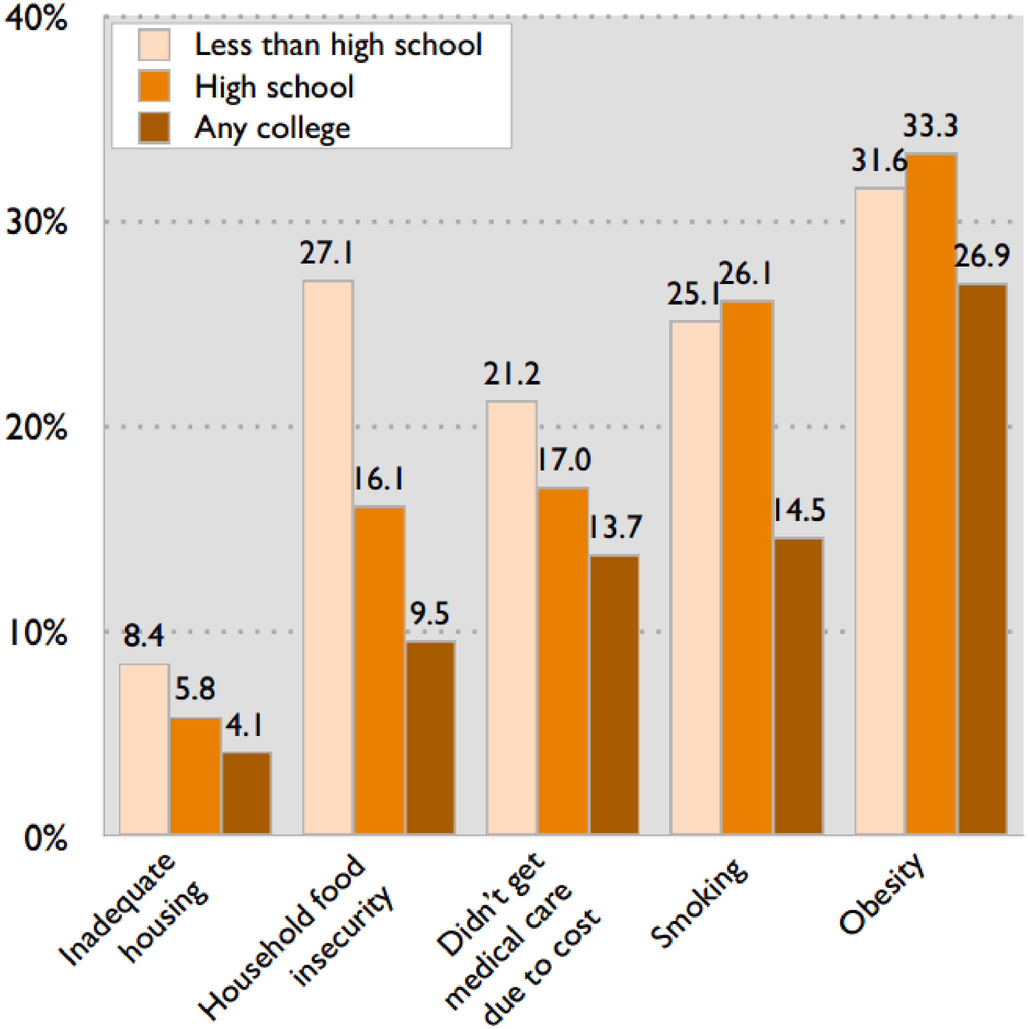

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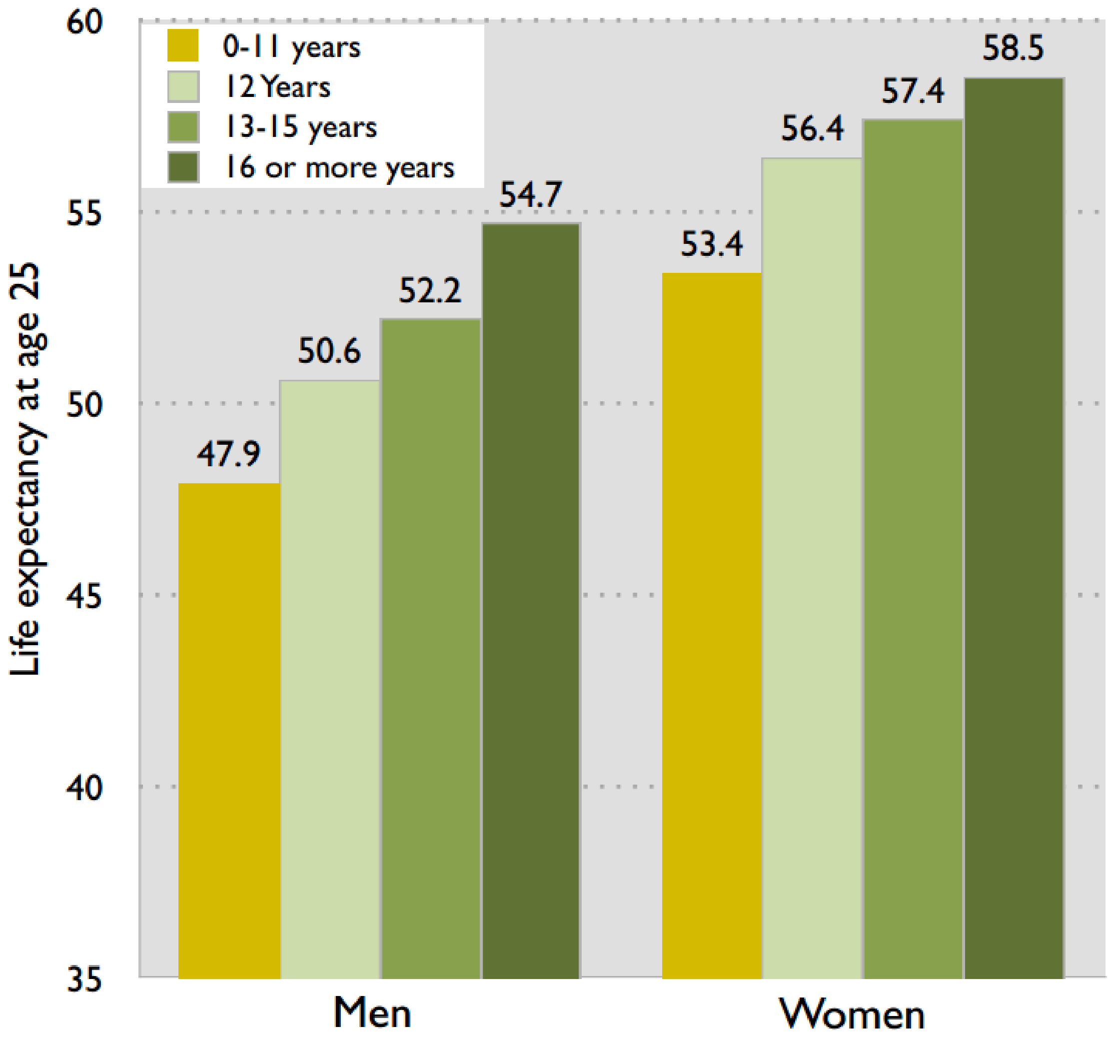

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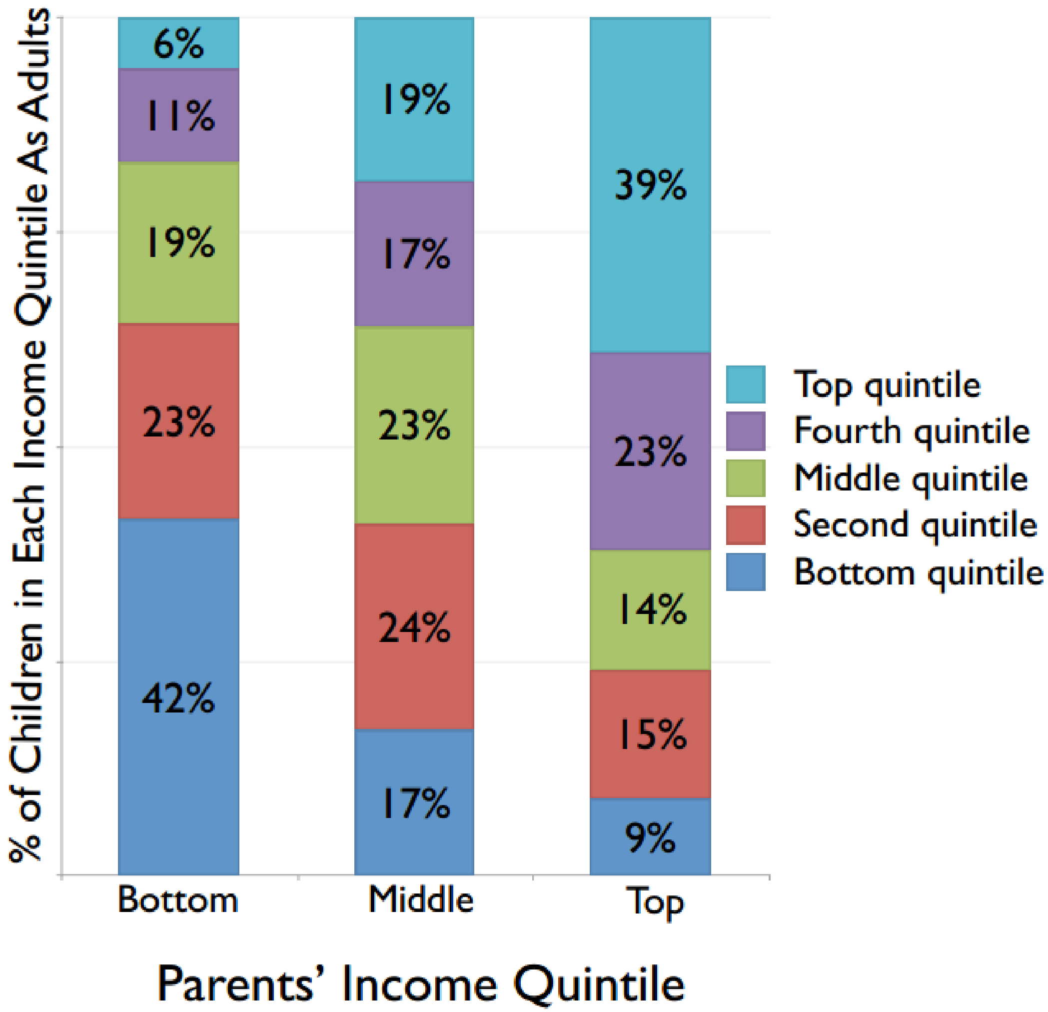

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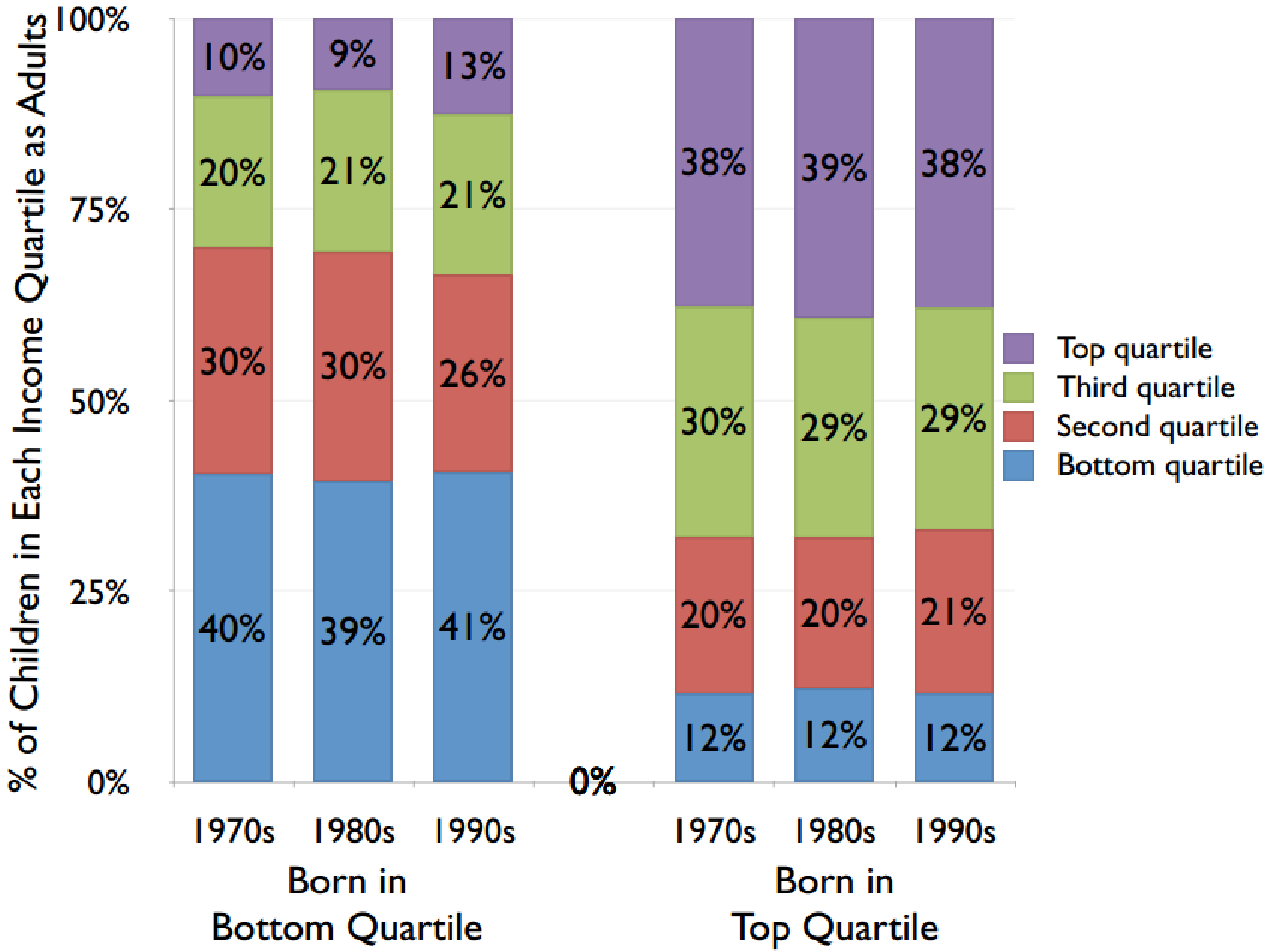

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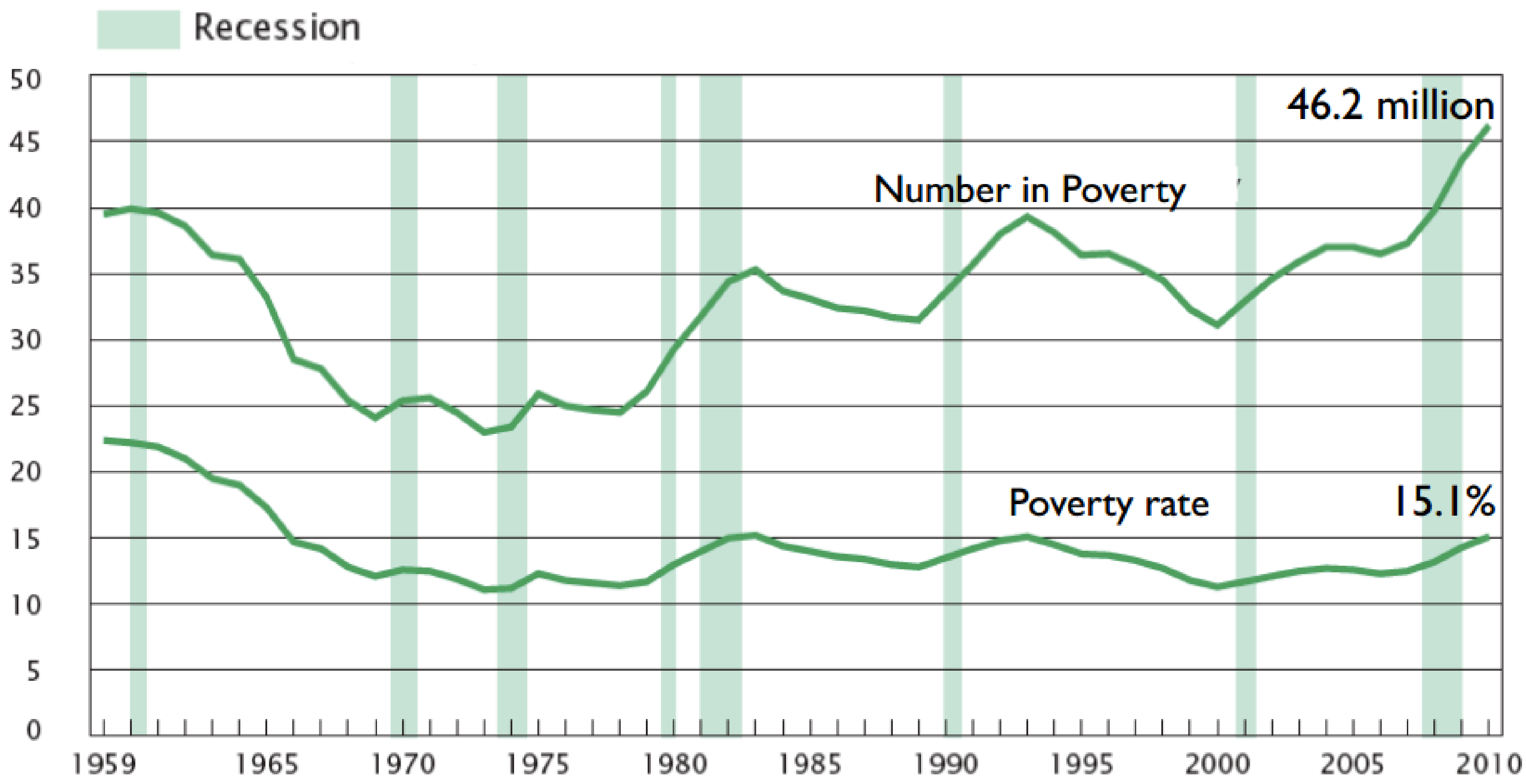

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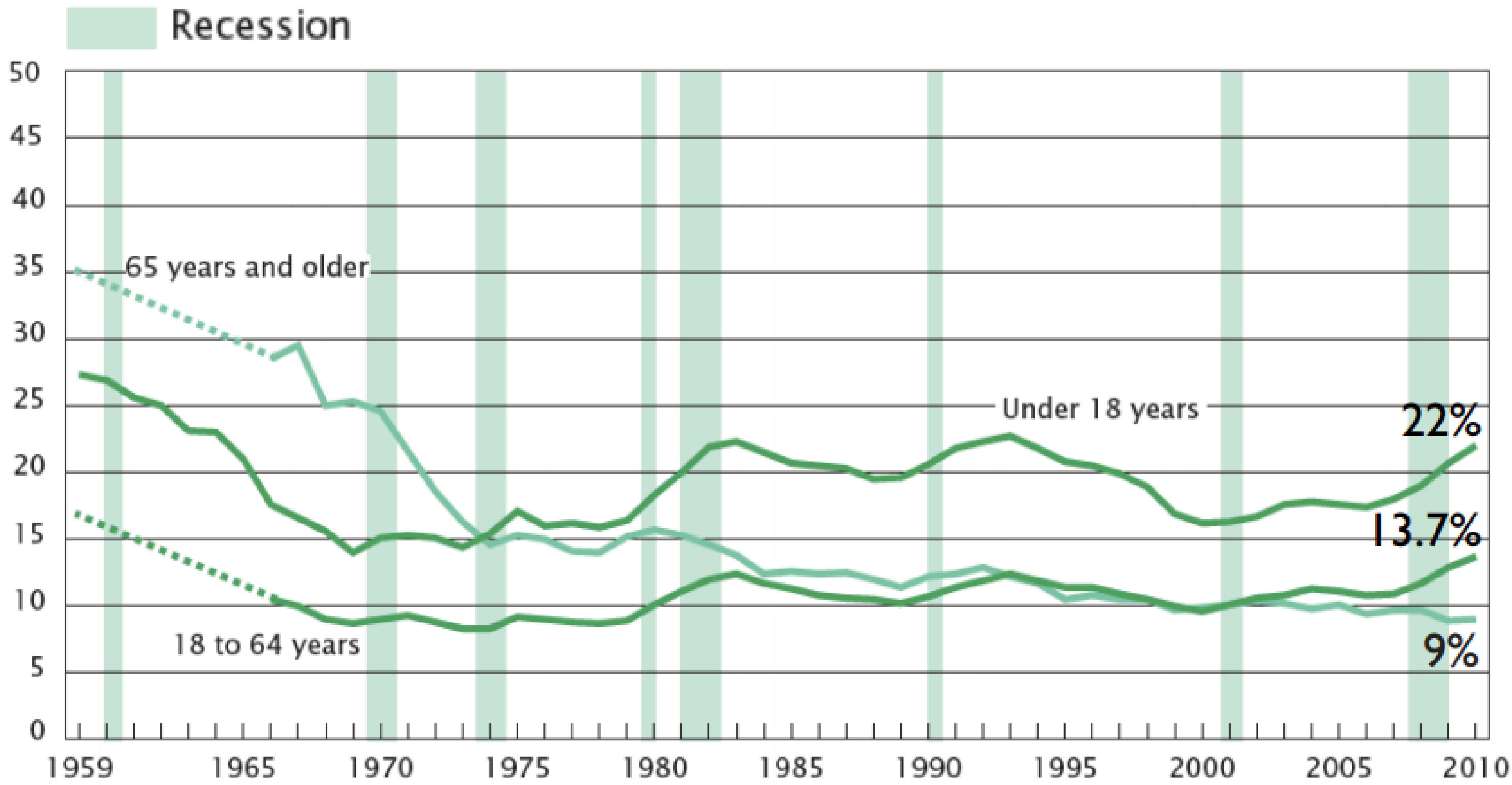

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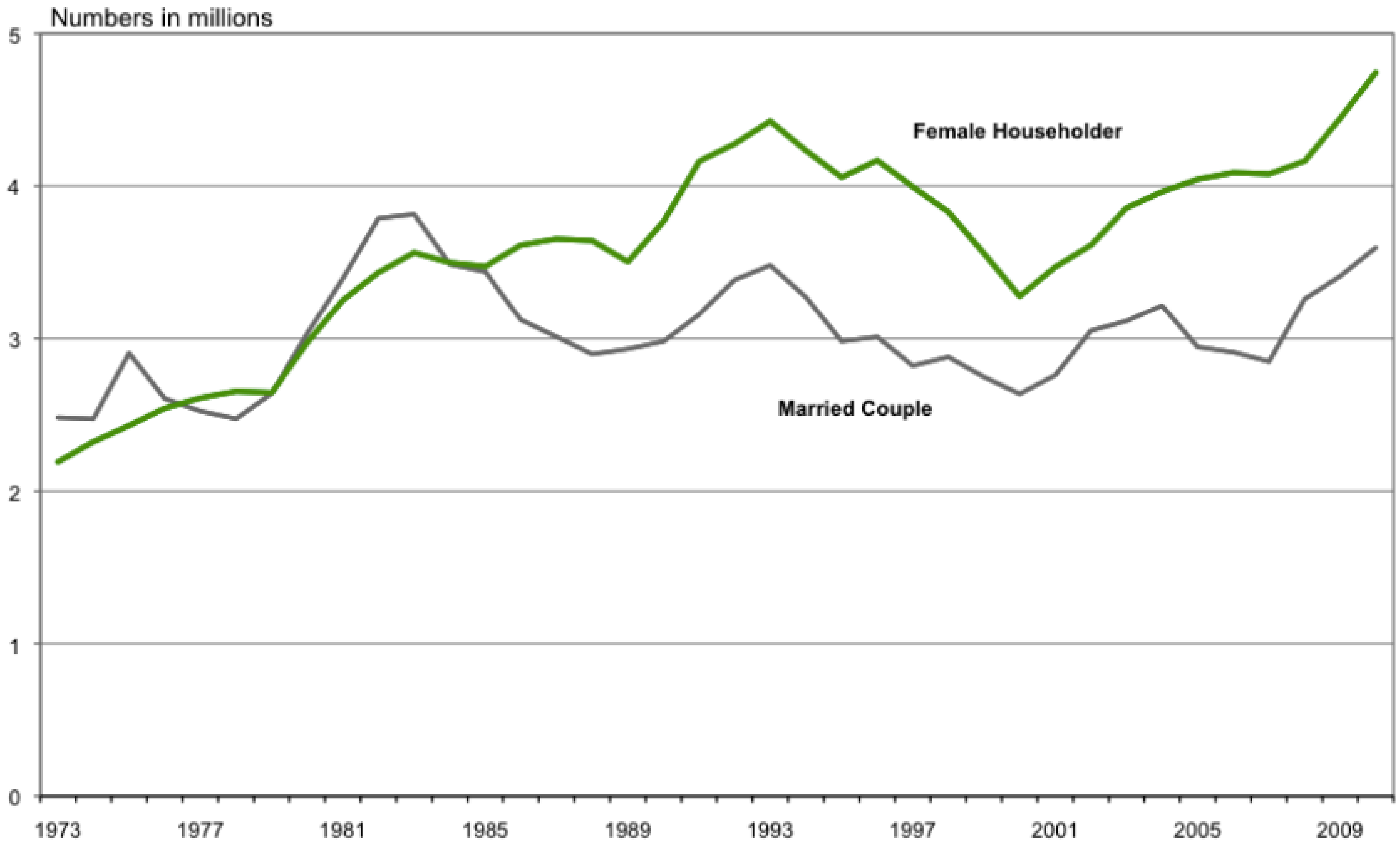

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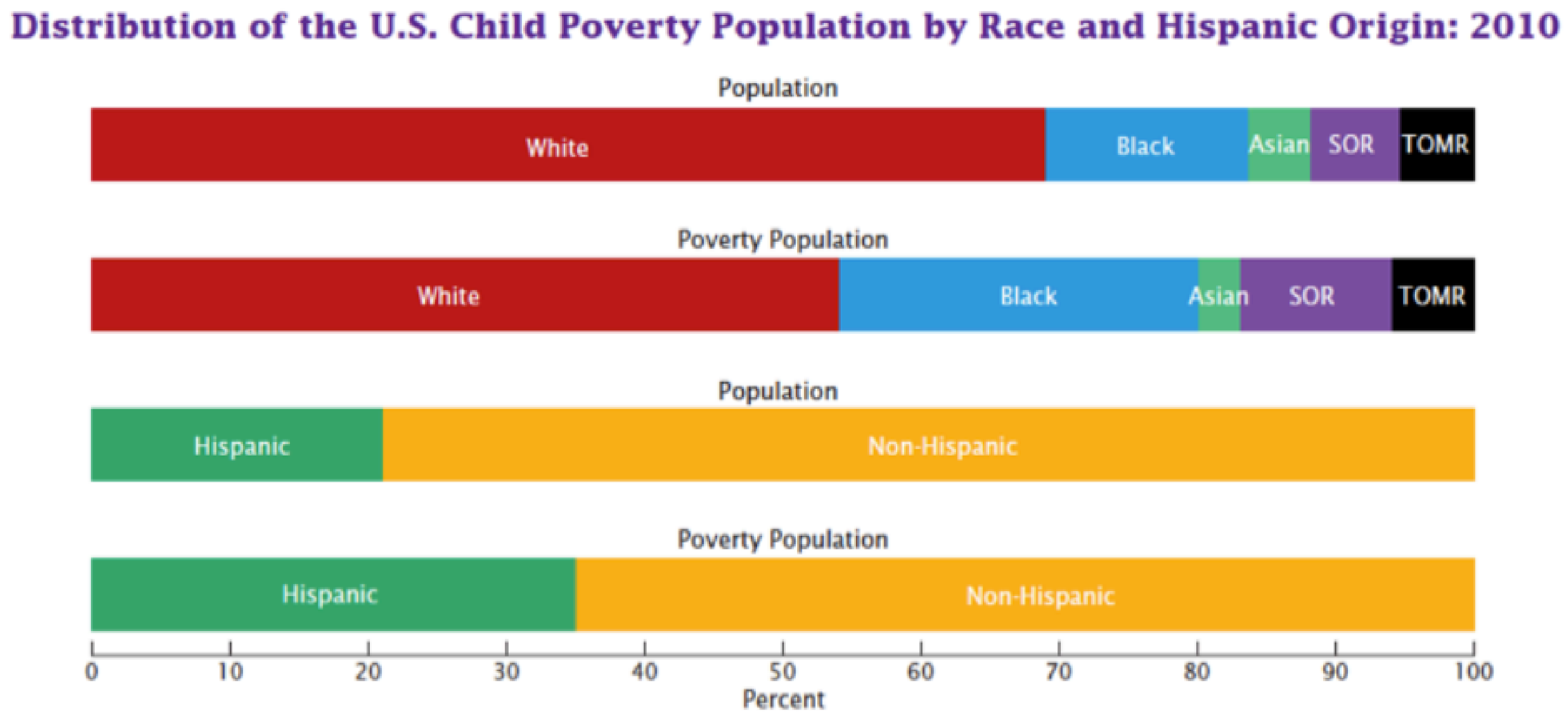

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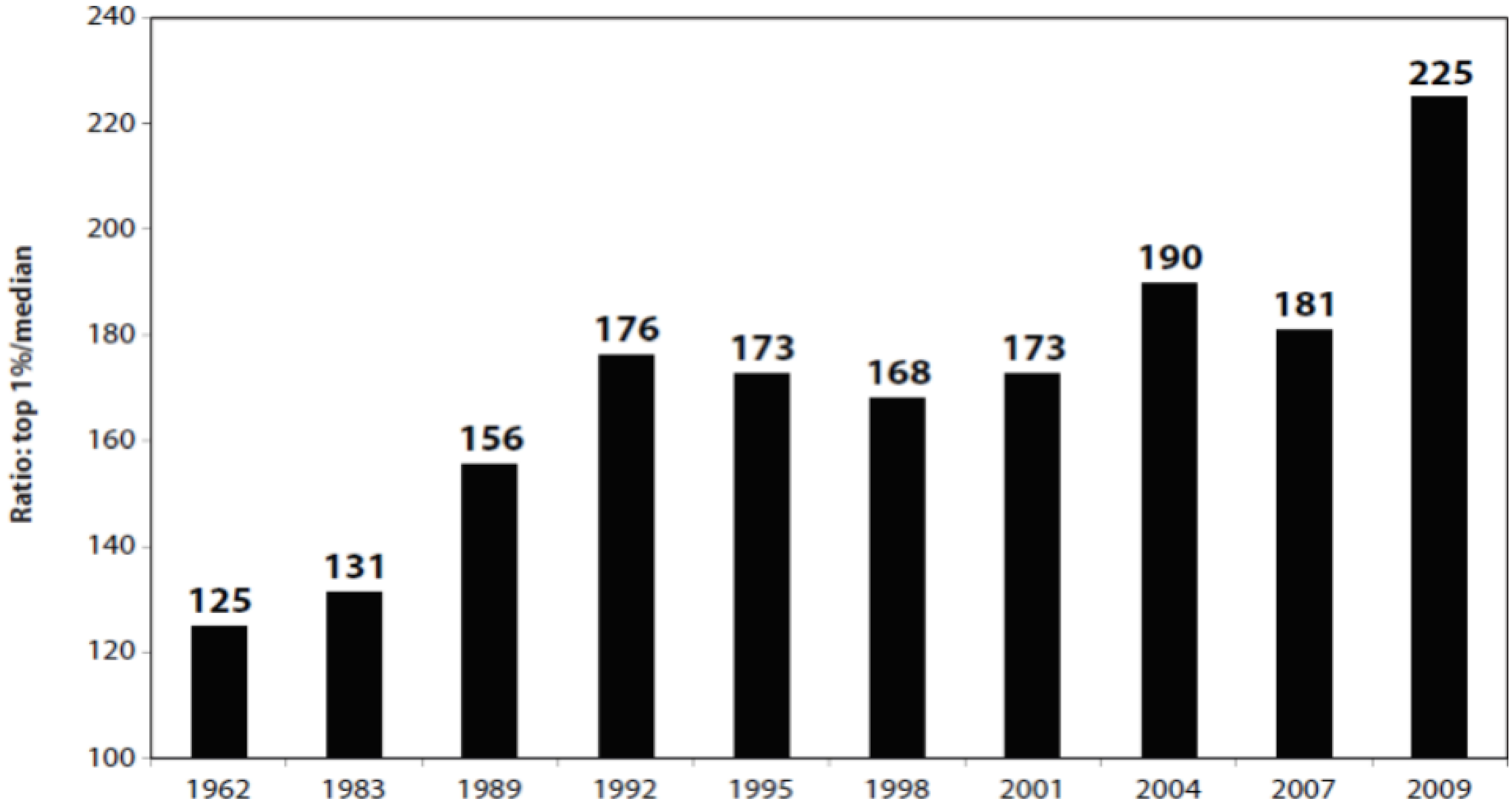


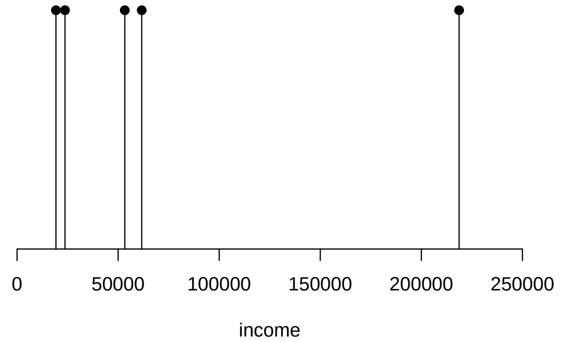
Modeling Income Distribution

Let $\{x_1, \cdots, x_n\}$ denote some sample. Then

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{1}{n} x_i$$

This can be used when we have census data.

```
load(url("http://freakonometrics.free.fr/
    income_5.RData"))
income <- sort(income)
plot(1:5, income)
```



It is possible to use survey data. If π_i denote the probability to be drawn, use weights

$$\omega_i \propto \frac{1}{n\pi_i}$$

The weighted average is then

$$\overline{x}_\omega = \sum_{i=1}^{n} \frac{\omega_i}{\omega} x_i$$

where $\omega = \sum \omega_i$. This is an unbaised estimator of the population mean.

Sometime, data are obtained from stratified samples : before sampling, members of the population are groupes in homogeneous subgroupes (called a strata).
Given S strata, such that the population in strata s is N_s, then

$$\overline{x}_S = \sum_{s=1}^{S} \frac{N_s}{N} \overline{x}_s \text{ where } \overline{x}_s = \frac{1}{N_s} \sum_{i \in \mathcal{S}_s} x_i$$

Statistical Tools Used to Describe the Distribution

Consider a sample $\{x_1, \cdots, x_n\}$. Usually, the order is not important. So let us order those values,

$$\underbrace{x_{1:n}}_{\min\{x_i\}} \leq x_{2:n} \leq \cdots \leq x_{n-1:n} \leq \underbrace{x_{n:n}}_{\max\{x_i\}}$$

As usual, assume that x_i's were randomly drawn from an (unknown) distribution F.

If F denotes the cumulative distribution function, $F(x) = \mathbb{P}(X \leq x)$, one can prove that

$$F(x_{i:n}) = \mathbb{P}(X \leq x_{i:n}) \sim \frac{i}{n}$$

The quantile function is defined as the inverse of the cumulative distribution function F,

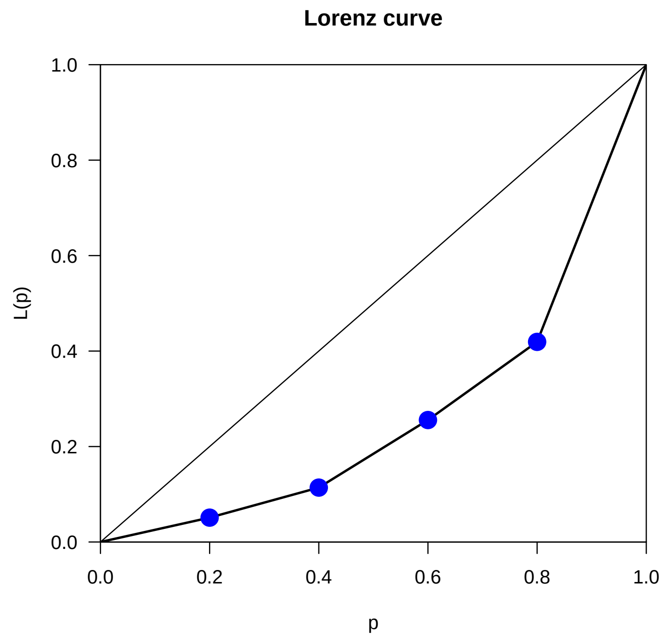
$$Q(u) = F^{-1}(u) \text{ or } F(Q(u)) = \mathbb{P}(X \leq Q(u)) = u$$

Lorenz curve

The empirical version of Lorenz curve is

$$L = \left\{ \frac{i}{n}, \frac{1}{n\overline{x}} \sum_{j \leq i} x_{j:n} \right\}$$

```
> plot((0:5)/5,c(0,cumsum(income)/sum(income
    )))
```



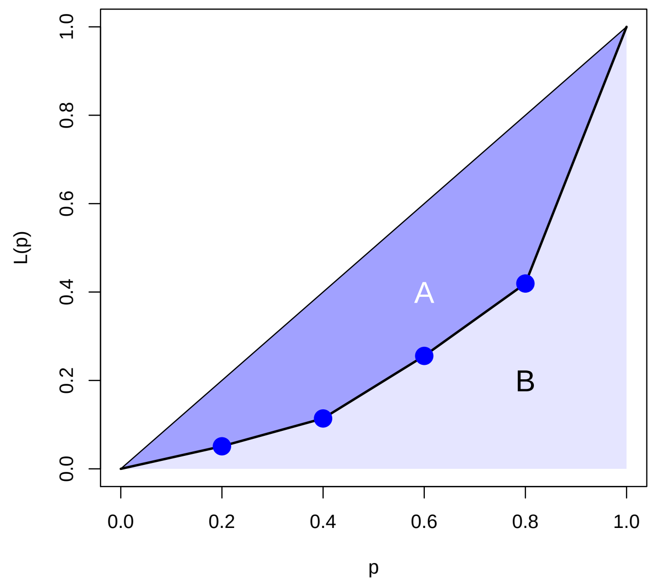
Gini Coefficient

Gini coefficient is defined as the ratio of areas, $\dfrac{\mathsf{A}}{\mathsf{A}+\mathsf{B}}$.

It can be defined using order statistics as

$$G = \frac{2}{n(n-1)\overline{x}} \sum_{i=1}^{n} i \cdot x_{i:n} - \frac{n+1}{n-1}$$

```
> n <- length(income)
> mu <- mean(income)
> 2*sum((1:n)*sort(income))/ (mu*n*(n-1))-(n
    +1)/(n-1)
[1] 0.5800019
```



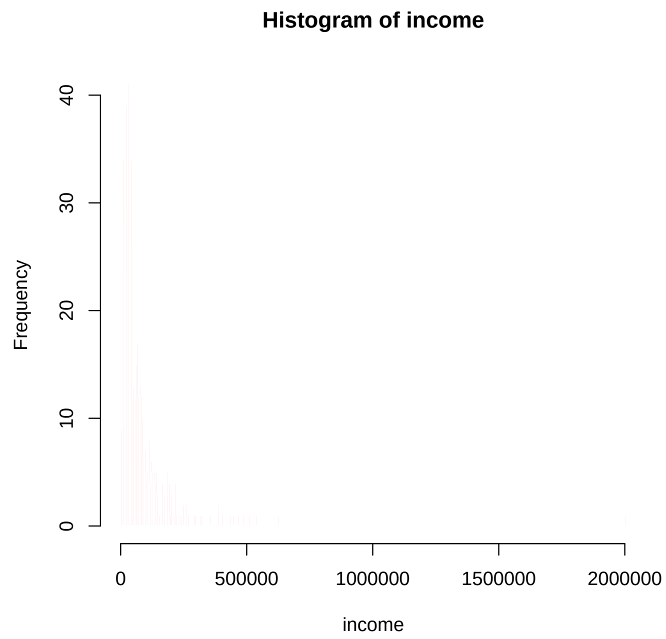
Distribution Fitting

Assume that we now have more observations,

```
> load(url("http://freakonometrics.free.fr/income_500.RData"))
```

We can use some histogram to visualize the distribution of the income

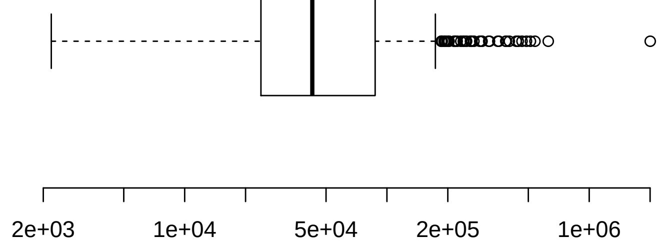
```
> summary(income)
    Min.  1st Qu.   Median     Mean  3rd Qu.
     Max.
    2191    23830    42750    77010    87430
     2003000
> sort(income)[495:500]
[1]  465354  489734  512231  539103  627292
     2003241
> hist(income,breaks=seq(0,2005000,by=5000))
```

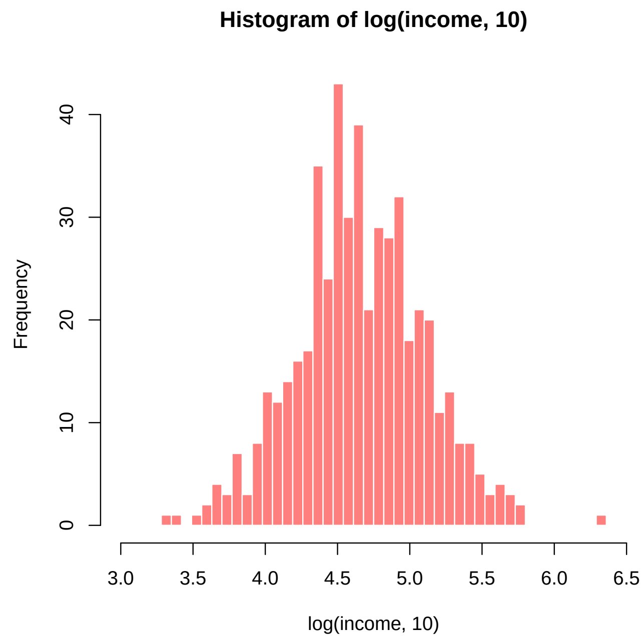


Distribution Fitting

Because of the dispersion, look at the histogram of the logarithm of the data

```
> hist(log(income,10),breaks=seq(3,6.5,
    length=51))
> boxplot(income,horizontal=TRUE,log="x")
```

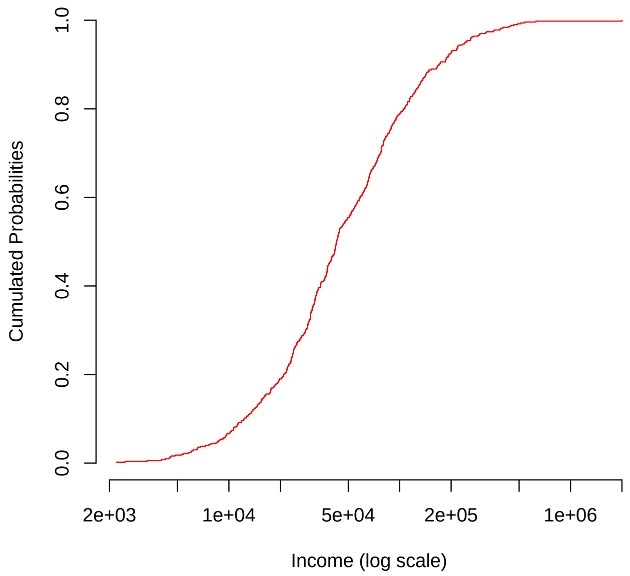




Distribution Fitting

The cumulative distribution function (on the log of the income)

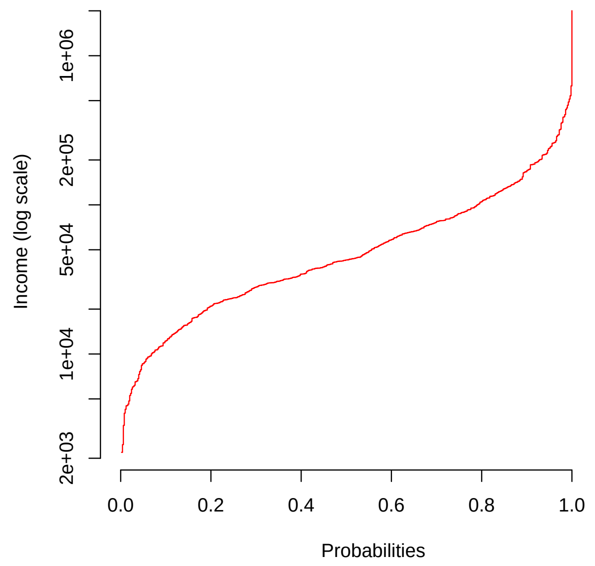
```
> u <- sort(income)
> v <- (1:500)/500
> plot(u,v,type="s",log="x")
```



Distribution Fitting

If we invert that graph, we have the quantile function

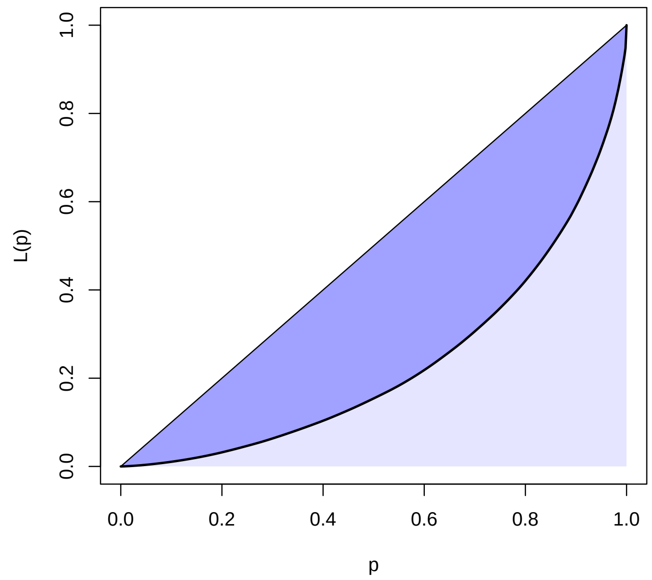
```
> plot(v,u,type="s",col="red",log="y")
```



Distribution Fitting

On that dataset, Lorenz curve is

```
> plot((0:500)/500,c(0,cumsum(income)/sum(
    income)))
```



Distribution and Confidence Intervals

There are two techniques to get the distribution of an estimator $\widehat{\theta}$,

- a parametric one, based on some assumptions on the underlying distribution,
- a nonparametric one, based on sampling techniques

If X_i's have a $\mathcal{N}(\mu, \sigma^2)$ distribution, then $\overline{X} \sim \mathcal{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$

But sometimes, distribution can only be obtained as an approximation, because of asymptotic properties.

From the central limit theorem, $\overline{X} \rightarrow \mathcal{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ as $n \rightarrow \infty$.

In the nonparametric case, the idea is to generate pseudo-samples of size n, by resampling from the original distribution.

Bootstraping

Consider a sample $\boldsymbol{x} = \{x_1, \cdots, x_n\}$. At step $b = 1, 2, \cdots, B$, generate a pseudo sample $\boldsymbol{x}^b$ by sampling (with replacement) within sample $\boldsymbol{x}$. Then compute any statistic $\widehat{\theta}(\boldsymbol{x}^b)$

```
> boot <- function(sample,f,b=500){
+ F <- rep(NA,b)
+ n <- length(sample)
+ for(i in 1:b){
+ idx <- sample(1:n,size=n,replace=TRUE)
+ F[i] <- f(sample[idx])}
+ return(F)}
```

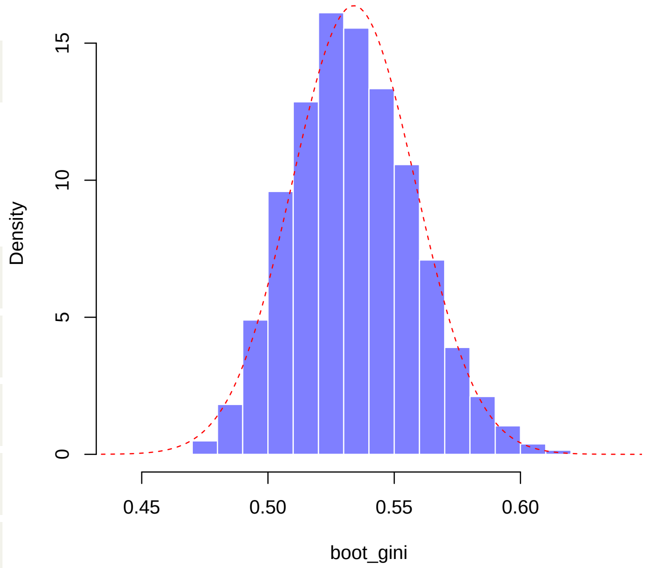
Bootstraping

Let us generate 10,000 bootstraped sample, and compute Gini index on those

```
>boot_gini <- boot(income,gini,1e4)
```

To visualize the distribution of the index

```
> hist(boot_gini,probability=TRUE)
> u <- seq(.4,.7,length=251)
> v <- dnorm(u,mean(boot_gini),sd(boot_gini)
    )
> lines(u,v,col="red",lty=2)
```



Continuous Versions

The empirical cumulative distribution function

$$\widehat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(x_i \leq x)$$

Observe that

$$\widehat{F}_n(x_{j:n}) = \frac{j}{n}$$

If F is absolutely continuous,

$$F(x) = \int_0^x f(t)dt \text{ i.e. } f(x) = \frac{dF(x)}{dx}.$$

Then

$$\mathbb{P}(x \in [a,b]) = \int_a^b f(t)dt = F(b) - F(a).$$

Continuous Versions

One can define quantiles as

$$x = Q(p) = F^{-1}(p)$$

The expected value is

$$\mu = \int_0^\infty x f(x) dx = \int_0^\infty [1 - F(x)] dx = \int_0^1 Q(p) dp.$$

We can compute the average standard of living of the group below z. This is equivalent to the expectation of a truncated distribution.

$$\mu_z^- = \frac{1}{F(z)} \int_0^z x f(x) dx = \int_0^\infty \left[1 - \frac{F(x)}{F(z)}\right] fx$$

Continuous Versions

Lorenz curve is $p \mapsto L(p)$ with

$$L(p) = \frac{1}{\mu}\int_0^{Q(p)} xf(x)dx$$

Gastwirth (1971) proved that

$$L(p) = \frac{1}{\mu}\int_0^{p} Q(u)du = \frac{\int_0^p Q(u)du}{\int_0^1 Q(u)du}$$

The numerator sums the incomes of the bottom p proportion of the population. The denominator sums the incomes of all the population.

L is a $[0,1] \rightarrow [0,1]$ function, continuous if F is continuous. Observe that L is increasing, since

$$\frac{dL(p)}{dp} = \frac{Q(p)}{\mu}$$

Further, L is convex

The sample case

$$L\left(\frac{i}{n}\right)=\frac{\sum_{j=1}^{i} x_{j:n}}{\sum_{j=1}^{n} x_{j:n}}$$

The points $\{i/n, L(i/n)\}$ are then linearly interpolated to complete the corresponding Lorenz curve.

The continuous distribution case

$$L(p)=\frac{\int_0^{F^{-1}(p)} ydF(y)}{\int_0^{\infty} ydF(y)}=\frac{1}{\mathbb{E}(X)}\int_0^p F^{-1}(u)du$$

with $p \in (0,1)$.

Let L be a continuous function on $[0,1]$, then L is a Lorenz curve if and only if

$$L(0)=0,\ L(1)=1,\ L'(0^+)\geq 0 \text{ and } L''(p)\geq 0 \text{ on } [0,1].$$

From Lorenz to Bonferroni

The Bonferroni curve is

$$B(p) = \frac{L(p)}{p}$$

and the Bonferroni index is

$$BI = 1 - \int_0^1 B(p)dp.$$

Define

$$P_i = \frac{i}{n} \text{ and } Q_i = \frac{1}{n\overline{x}} \sum_{j=1}^{i} x_j$$

then

$$B = \frac{1}{n-1} \sum_{i=1}^{n-1} \left(\frac{P_i - Q_i}{P_i} \right)$$

Gini and Pietra indices

The Gini index is defined as twice the area between the egalitarian line and the Lorenz curve

$$G = 2\int_0^1 [p - L(p)]dp = 1 - 2\int_0^1 L(p)dp$$

which can also be writen

$$1 - \frac{1}{\mathbb{E}(X)}\int_0^\infty [1 - F(x)]^2 dx$$

Pietra index is defined as the maximal vertical deviation between the Lorenz curve and the egalitarian line

$$P = \max_{p\in(0,1)} \{p - L(p)\} = \frac{\mathbb{E}(|X - \mathbb{E}(X)|)}{2\mathbb{E}(X)}$$

if F is strictly increasing (the maximum is reached in $p^\star = F(\mathbb{E}(X))$)

Examples

E.g. consider the uniform distribution

$$F(x) = \min\{1, \frac{x-a}{b-a}\mathbf{1}(x \geq a)\}$$

Then

$$L(p) = \frac{2ap + (b-a)^2p^2}{a+b}$$

and Gini index is

$$G = \frac{b-a}{3(a+b)}$$

E.g. consider a Pareto distribution,

$$F(x) = 1 - \left(\frac{x_0}{x}\right)^\alpha,\ x \geq x_0,$$

with shape parameter $\alpha > 0$. Then

$$F^{-1}(u) = \frac{x_0}{(1-u)^{\frac{1}{\alpha}}}$$

and

$$L(p) = 1 - [1-p]^{1-\frac{1}{\alpha}} \ p \in (0,1).$$

and Gini index is

$$G = \frac{1}{2\alpha - 1}$$

while Pietra index is, if $\alpha > 1$

$$P = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}$$

E.g. consider the lognormal distribution,

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right)$$

then

$$L(p) = \Phi(\Phi^{-1}(p) - \sigma) \ p \in (0,1).$$

and Gini index is

$$G = 2\Phi\left(\frac{\sigma}{\sqrt{2}}\right) - 1$$

Fitting a Distribution

The standard technique is based on maximum likelihood estimation, provided by

```
> library(MASS)
> fitdistr(income,"lognormal")
     meanlog          sdlog
  10.72264538    1.01091329
 ( 0.04520942) ( 0.03196789)
```

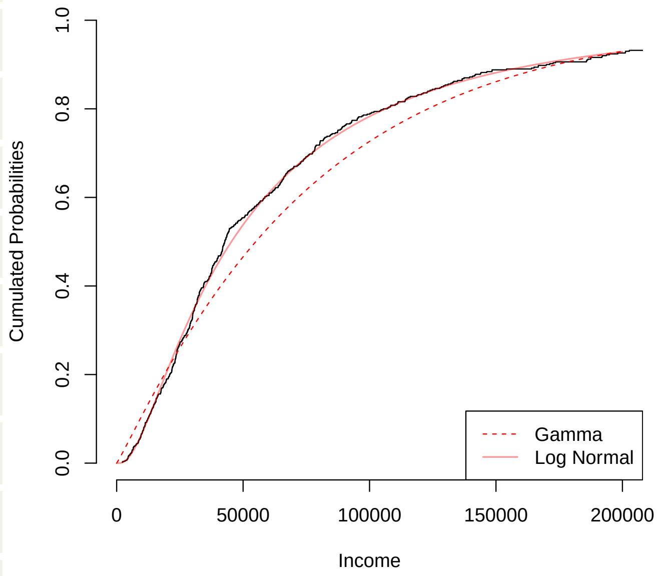
For other distribution (such as the Gamma distribution), we might have to rescale

```
> (fit_g <- fitdistr(income/1e2,"gamma"))
       shape              rate
  1.0812757769    0.0014040438
 (0.0473722529) (0.0000544185)
> (fit_ln <- fitdistr(income/1e2,"lognormal"))
     meanlog          sdlog
  6.11747519     1.01091329
 (0.04520942) (0.03196789)
```

Fitting a Distribution

We can compare the densities

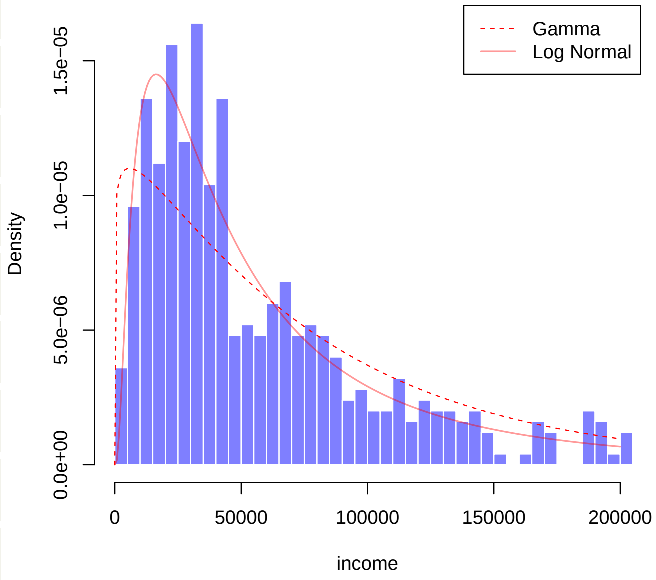
```
> u=seq(0,2e5,length=251)
> hist(income,breaks=seq(0,2005000,by=5000),
   col=rgb(0,0,1,.5),border="white",xlim=c
   (0,2e5),probability=TRUE)
> v_g <- dgamma(u/1e2, fit_g$estimate[1],fit
   _g$estimate[2])/1e2
> v_ln <- dlnorm(u/1e2, fit_ln$estimate[1],
   fit_ln$estimate[2])/1e2
> lines(u,v_g,col="red",lty=2)
> lines(u,v_ln,col=rgb(1,0,0,.4))
```



Fitting a Distribution

or the cumulative distributions

```
x <- sort(income)
y <- (1:500)/500
plot(x,y,type="s",col="black")
v_g <- pgamma(u/1e2, fit_g$estimate[1],fit_g
    $estimate[2])
v_ln <- plnorm(u/1e2, fit_ln$estimate[1],fit
    _ln$estimate[2])
lines(u,v_g,col="red",lty=2)
lines(u,v_ln,col=rgb(1,0,0,.4))
```



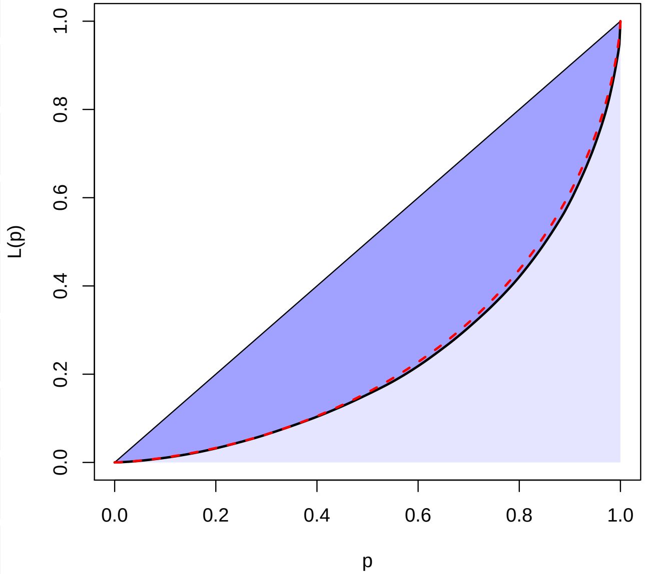
One might consider the parametric version of Lorenz curve, to confirm the goodness of fit, e.g. a lognormal distribution with $\sigma = 1$ since

```
> fitdistr(income,"lognormal")
     meanlog        sdlog
  10.72264538    1.01091329
```

Fitting a Distribution

We can use functions of R

```
library(ineq)
Lc.sim <- Lc(income)
plot(0:1,0:1,xlab="p",ylab="L(p)",col="white
    ")
polygon(c(0,1,1,0),c(0,0,1,0),col=rgb
    (0,0,1,.1),border=NA)
polygon(Lc.sim$p,Lc.sim$L,col=rgb(0,0,1,.3),
    border=NA)
lines(Lc.sim)
segments(0,0,1,1)
lines(Lc.lognorm, parameter=1,lty=2)
```



Standard Parametric Distribution

For those distributions, we mention the R names in the gamlss package. Inference can be done using

```
fit <-gamlss(y~1, family=LNO)
```

- **log normal**

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}}\, e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}},\ x \geq 0$$

with mean $e^{\mu+\sigma^2/2}$, median e^{μ}, and variance $(e^{\sigma^2}-1)e^{2\mu+\sigma^2}$

```
LNO(mu.link = "identity", sigma.link = "log")
dLNO(x, mu = 1, sigma = 0.1, nu = 0, log = FALSE)
```

- **gamma**

$$f(x) = \frac{x^{1/\sigma^2 - 1}\exp[-x/(\sigma^2\mu)]}{(\sigma^2\mu)^{1/\sigma^2}\Gamma(1/\sigma^2)},\ x \geq 0$$

with mean μ and variance σ^2

```
GA(mu.link = "log", sigma.link ="log")
dGA(x, mu = 1, sigma = 1, log = FALSE)
```

- **Pareto**

$$f(x) = \frac{\alpha\, x_\mathrm{m}^\alpha}{x^{\alpha+1}} \text{ for } x \geq x_m$$

with cumulated distribution

$$F(x) = 1 - \left(\frac{x_\mathrm{m}}{x}\right)^\alpha \text{ for } x \geq x_m$$

with mean $\dfrac{\alpha x_m}{(\alpha-1)}$ if $\alpha > 1$, and variance $\dfrac{x_\mathrm{m}^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2$.

```
PARETO2(mu.link = "log", sigma.link = "log")
dPARETO2(x, mu = 1, sigma = 0.5, log = FALSE)
```

Larger Families

- **GB1** - generalized Beta type 1

$$f(x) = \frac{|a|x^{ap-1}(1-(x/b)^a)^{q-1}}{b^{ap}B(p,q)},\ 0 < x^a < b^a$$

where b , p , and q are positive

```
GB1(mu.link = "logit", sigma.link = "logit", nu.link = "log",tau.link
     = "log")
dGB1(x, mu = 0.5, sigma = 0.4, nu = 1, tau = 1, log = FALSE)
```

The GB1 family includes the generalized gamma(GG), and Pareto as special cases.

- **GB2** - generalized Beta type 2

$$f(x) = \frac{|a|x^{ap-1}}{b^{ap}B(p,q)(1+(x/b)^a)^{p+q}}$$

```
GB2(mu.link = "log", sigma.link = "identity", nu.link = "log", tau.
    link = "log")
dGB2(x, mu = 1, sigma = 1, nu = 1, tau = 0.5, log = FALSE)
```

The GB2 nests common distributions such as the generalized gamma (GG), Burr, lognormal, Weibull, Gamma, Rayleigh, Chi-square, Exponential, and the log-logistic.

- Generalized Gamma

$$f(x) = \frac{(p/a^d)x^{d-1}e^{-(x/a)^p}}{\Gamma(d/p)},$$

Dealing with Binned Data

```
> load(url("http://freakonometrics.free.fr/income_binned.RData"))
> head(income_binned)
    low  high number  mean std_err
1     0  4999     95  3606     964
2  5000  9999    267  7686    1439
3 10000 14999    373 12505    1471
4 15000 19999    350 17408    1368
5 20000 24999    329 22558    1428
6 25000 29999    337 27584    1520
> tail(income_binned)
      low   high number   mean std_err
46 225000 229999     10 228374    1197
47 230000 234999     13 232920    1370
48 235000 239999     11 236341    1157
49 240000 244999     14 242359    1474
50 245000 249999     11 247782    1487
51 250000    Inf    228 395459  189032
```

Dealing with Binned Data

There is a dedicated package to work with such datasets,

```
> library(binequality)
```

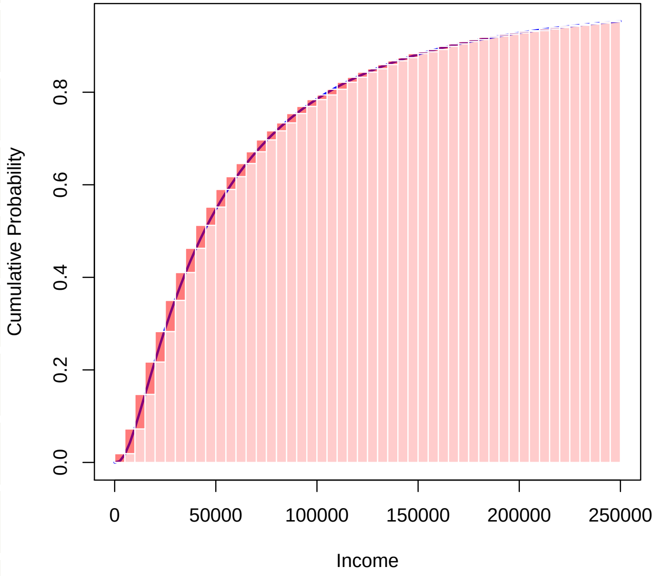
To fit a parametric distribution, e.g. a log-normal distribution, use functions of R

```
> n <- nrow(income_binned)
> fit_LN <- fitFunc(ID=rep("Fake Data",n), hb=income_binned[,"number"
    ], bin_min=income_binned[,"low"], bin_max=income_binned[,"high"],
    obs_mean=income_binned[,"mean"], ID_name="Country", distribution=
    LNO, distName="LNO")
Time difference of 0.09900618 secs
for LNO fit across 1 distributions
```

Dealing with Binned Data

To visualize the cumulated distribution function, use

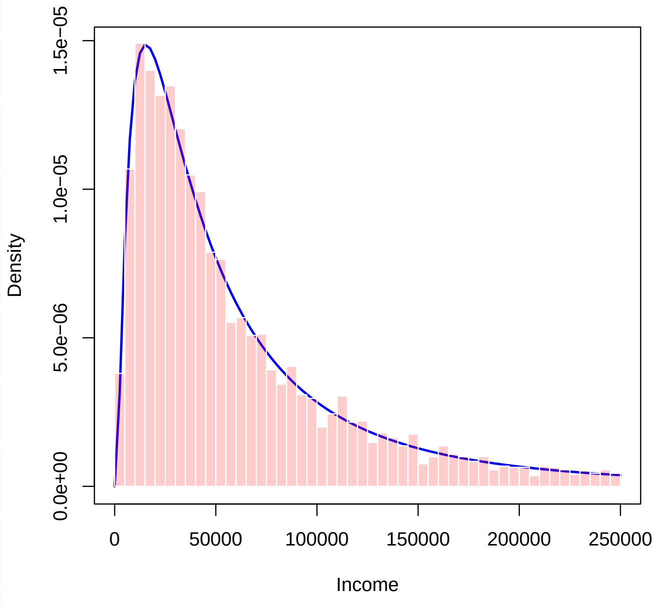
```
> N <- income_binned$number
> y1 <- cumsum(N)/sum(N)
> u <- seq(min(income_binned$low),max(income_binned$low),length=101)
> v <- plnorm(u,fit_LN$parameters[1],fit_LN$parameters[2])
> plot(u,v,col="blue",type="l",lwd=2,xlab="Income",ylab="Cumulative Probability")
> for(i in 1:(n-1)) rect(income_binned$low[i],0,income_binned$high[i],y1[i],col=rgb(1,0,0,.2))
> for(i in 1:(n-1)) rect(income_binned$low[i],y1[i],income_binned$high[i],c(0,y1)[i],col=rgb(1,0,0,.4))
```



Dealing with Binned Data

and to visualize the cumulated distribution function, use

```
> N=income_binned$number
> y2=N/sum(N)/diff(income_binned$low)
> u=seq(min(income_binned$low),max(income_
    binned$low),length=101)
> v=dlnorm(u,fit_LN$parameters[1],fit_LN$
    parameters[2])
> plot(u,v,col="blue",type="l",lwd=2,xlab="
    Income",ylab="Density")
> for(i in 1:(n-1)) rect(income_binned$low[i
    ],0,income_binned$high[i],y2[i],col=rgb
    (1,0,0,.2),border="white")
```



Dealing with Binned Data

But it is also possible to estimate *all* GB-distributions at once,

```
> fits=run_GB_family(ID=rep("Fake Data",n),hb=income_binned[,"number"
    ],bin_min=income_binned[,"low"],bin_max=income_binned[,"high"],obs
    _mean=income_binned[,"mean"],
+ ID_name="Country")
Time difference of 0.03800201 secs
for GB2 fit across 1 distributions

Time difference of 0.3090181 secs
for GG fit across 1 distributions

Time difference of 0.864049 secs
for BETA2 fit across 1 distributions
```

...

```
Time difference of 0.04900193 secs
```

```
for LOGLOG fit across 1 distributions

Time difference of 1.865106 secs
for PARETO2 fit across 1 distributions
```

```
> fits$fit.filter[,c("gini","aic","bic")]
        gini      aic      bic
1         NA       NA       NA
2   5.054377 34344.87 34364.43
3   5.110104 34352.93 34372.48
4         NA 53638.39 53657.94
5   4.892090 34845.87 34865.43
6   5.087506 34343.08 34356.11
7   4.702194 34819.55 34832.59
8   4.557867 34766.38 34779.41
9         NA 58259.42 58272.45
10  5.244332 34805.70 34818.73
```

```
> fits$best_model$aic
```

```
    Country obsMean distribution  estMean        var
1 Fake Data      NA          LNO 72328.86 6969188937
        cv   cv_sqr     gini     theil       MLD
1 1.154196 1.332168 5.087506 0.4638252 0.4851275
       aic      bic didConverge logLikelihood nparams
1 34343.08 34356.11        TRUE     -17169.54       2
    median       sd
1 44400.23 83481.67
```

That was easy, those were simulated data...

Dealing with Binned Data

Consider now some real data,

```
> data = read.table("http://freakonometrics.free.fr/us_income.txt",
    sep=",",header=TRUE)
> head(data)
    low  high  number_1000s  mean  std_err
1     0  4999          4245  1249       50
2  5000  9999          5128  7923       30
3 10000 14999          7149 12389       28
4 15000 19999          7370 17278       26
> tail(data)
       low     high  number_1000s   mean  std_err
39  190000   194999           361  192031      115
40  195000   199999           291  197120      135
41  200000   249999          2160  219379      437
42  250000  9999999          2498  398233     6519
```

Dealing with Binned Data

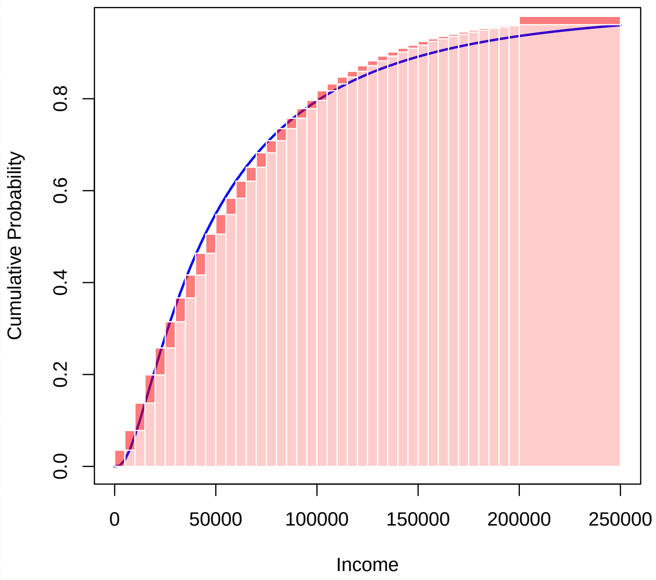
To fit a parametric distribution, e.g. a log-normal distribution, use

```
> n <- nrow(data)
> fit_LN <- fitFunc(ID=rep("US",n), hb=data[,"number_1000s"], bin_min=data[,"low"], bin_max=data[,"high"], obs_mean=data[,"mean"], ID_name="Country", distribution=LNO, distName="LNO")
Time difference of 0.1040058 secs
for LNO fit across 1 distributions
```

Dealing with Binned Data

To visualize the cumulated distribution function, use

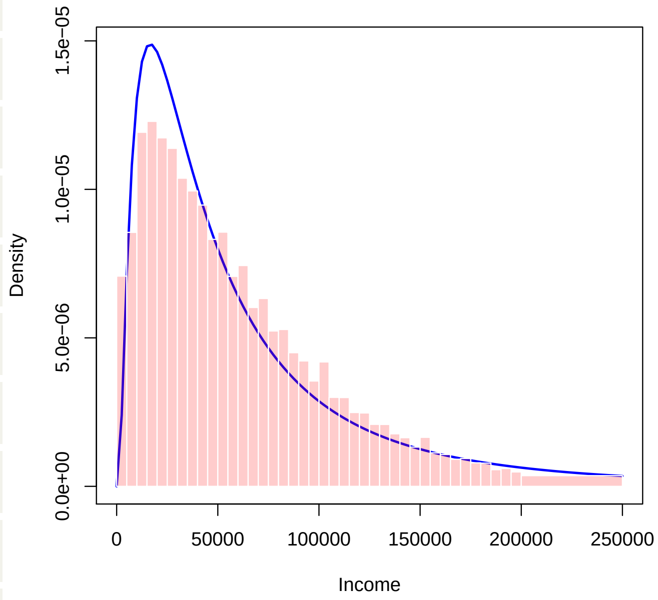
```
> N <- income_binned$number
> y1 <- cumsum(N)/sum(N)
> u <- seq(min(income_binned$low),max(income
    _binned$low),length=101)
> v <- plnorm(u,fit_LN$parameters[1],fit_LN$
    parameters[2])
> plot(u,v,col="blue",type="l",lwd=2,xlab="
    Income",ylab="Cumulative Probability")
> for(i in 1:(n-1)) rect(income_binned$low[i
    ],0,income_binned$high[i],y1[i],col=rgb
    (1,0,0,.2))
> for(i in 1:(n-1)) rect(income_binned$low[i
    ],y1[i],income_binned$high[i],c(0,y1)[i
    ],col=rgb(1,0,0,.4))
```



Dealing with Binned Data

and to visualize the cumulated distribution function, use

```
> N=income_binned$number
> y2=N/sum(N)/diff(income_binned$low)
> u=seq(min(income_binned$low),max(income_binned$low),length=101)
> v=dlnorm(u,fit_LN$parameters[1],fit_LN$parameters[2])
> plot(u,v,col="blue",type="l",lwd=2,xlab="Income",ylab="Density")
> for(i in 1:(n-1)) rect(income_binned$low[i],0,income_binned$high[i],y2[i],col=rgb(1,0,0,.2),border="white")
```



Dealing with Binned Data

And the winner is....

```
> fits$fit.filter[,c("gini","aic","bic")]
        gini      aic      bic
1   4.413411 825368.7 825407.4
2   4.395078 825598.8 825627.9
3   4.455112 825502.4 825531.5
4   4.480844 825881.5 825910.6
5   4.413282 825315.3 825344.4
6   4.922123 832408.2 832427.6
7   4.341085 827065.2 827084.6
8   4.318694 826112.9 826132.2
9         NA 831054.2 831073.6
10        NA       NA       NA
```

```
> fits$best_model$aic
  Country obsMean distribution  estMean        var
1      US      NA           GG 65147.54 3152161910
```

```
          cv     cv_sqr     gini     theil       MLD
1  0.8617995  0.7426984  4.395078  0.3251443  0.3904942
        aic        bic  didConverge  logLikelihood  nparams
1  825598.8  825627.9         TRUE      -412796.4        3
   median        sd
1  48953.6  56144.12
```

Inequality Comparisons (2-person Economy)

not much to say... any measure of dispersion is appropriate

— income gap $x_2 - x_1$

— proportional gap $\dfrac{x_2}{x_1}$

— any functional of the distance

$$\sqrt{|x_2 - x_1|}$$

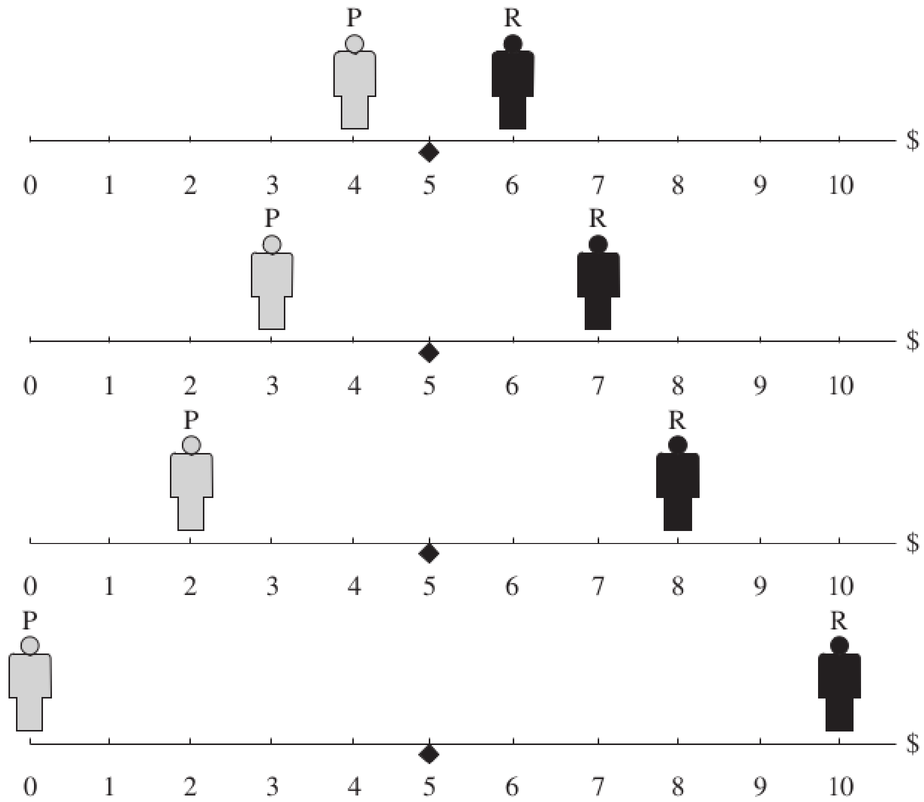
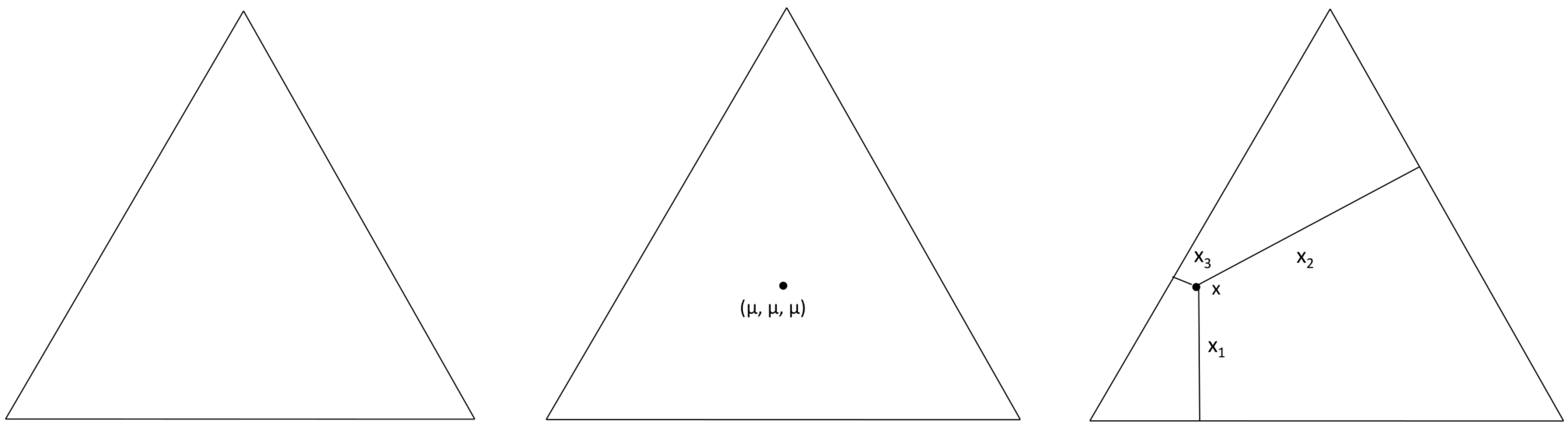


Figure 2.2. Inequality in a two-person world.

graphs are from Amiel & Cowell (1999, ebooks.cambridge.org)

Inequality Comparisons (3-person Economy)

Consider any 3-person economy, with incomes $\boldsymbol{x} = \{x_1, x_2, x_3\}$. This point can be visualized in Kolm triangle.



Inequality Comparisons (3-person Economy)

```
kolm=function(p=c(200,300,500)){
p1=p/sum(p)
y0=p1[2]
x0=(2*p1[1]+y0)/sqrt(3)
plot(0:1,0:1,col="white",xlab="",ylab="",
     axes=FALSE,ylim=c(0,1))
polygon(c(0,.5,1,0),c(0,.5*sqrt(3),0,0))
points(x0,y0,pch=19,col="red")}
```

Inequality Comparisons (n-person Economy)

In a n-person economy, comparison are clearly more difficult

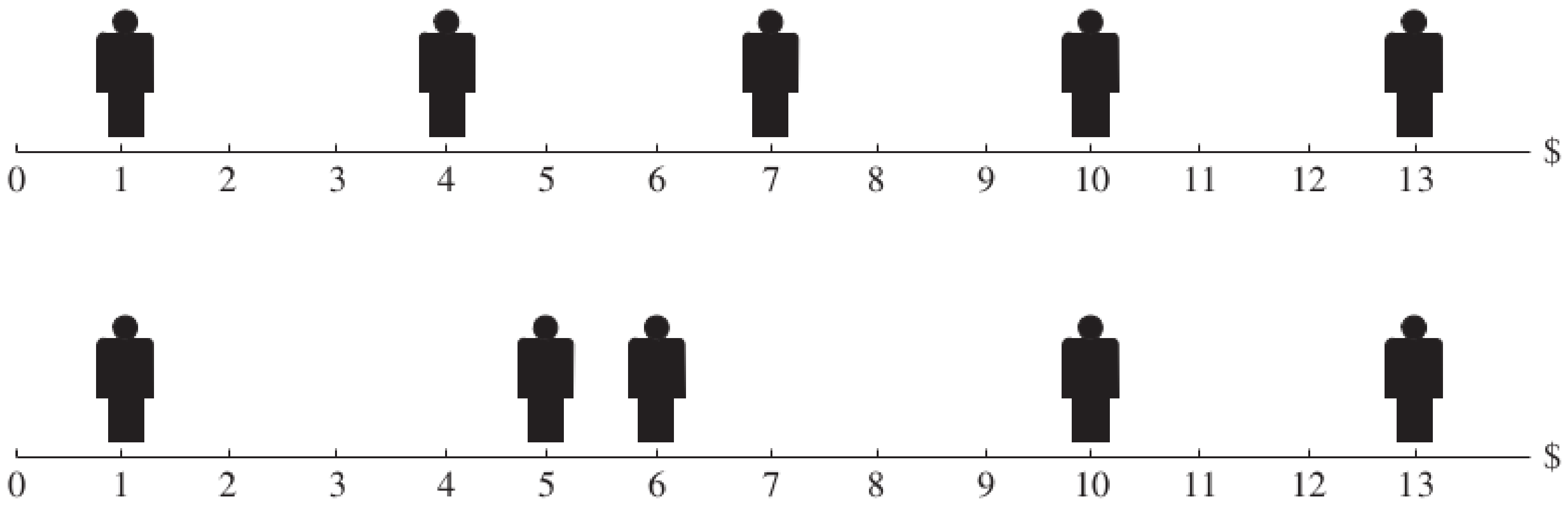


Figure 1.1. A simple distributional experiment.

Inequality Comparisons (n-person Economy)

Why not look at inequality per subgroups,

If we focus at the top of the distribution (same holds for the bottom),
→ rising inequality

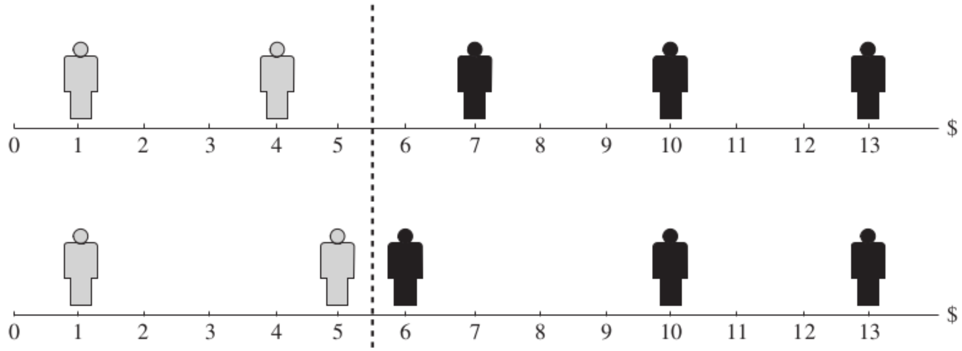


Figure 1.2. A simple distributional experiment: second view.

If we focus at the middle of the distribution,
→ falling inequality

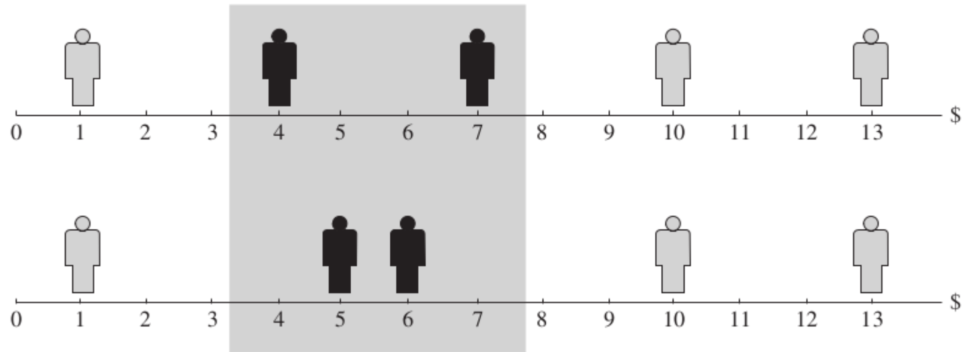


Figure 1.3. A simple distributional experiment: third view.

Inequality Comparisons (n-person Economy)

To measure inequality, we usually

- define 'equality' based on some reference point / distribution
- define a distance to the reference point / distribution
- aggregate individual distances

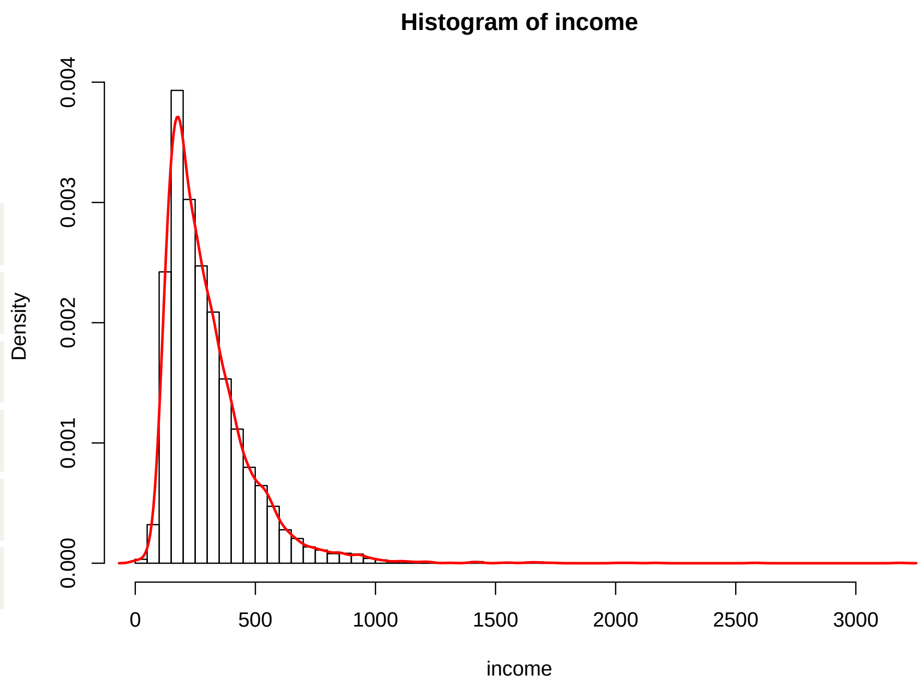
We want to visualize the distribution of incomes

```
> income <- read.csv("http://www.vcharite.univ-mrs.fr/pp/lubrano/
    cours/fes96.csv",sep=";",header=FALSE)$V1
```

$$F(x) = \mathbb{P}(X \leq x) = \int_0^x f(t)dt$$

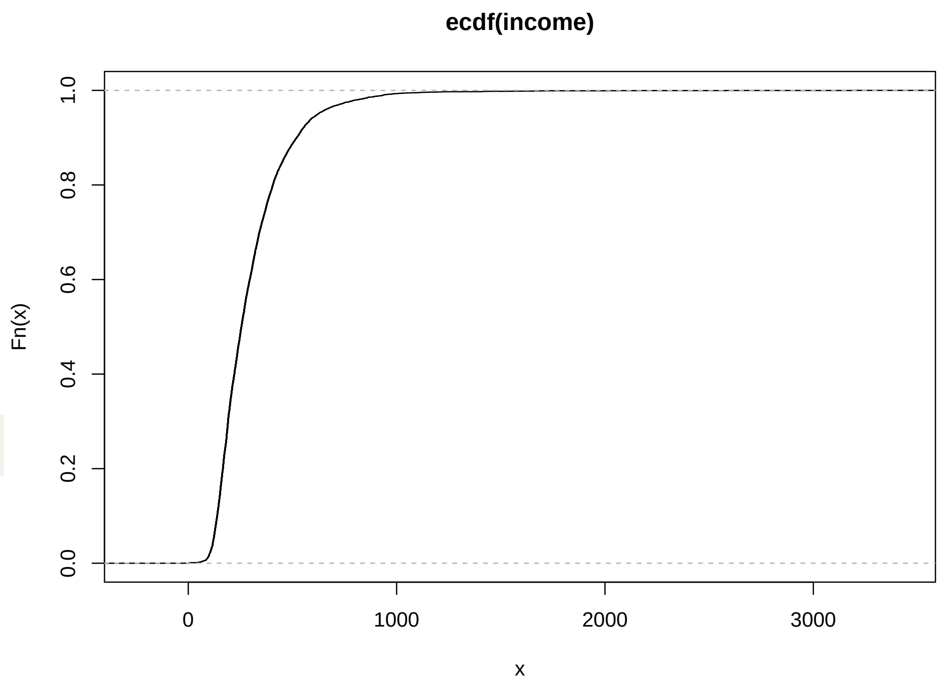
Densities are usually difficult to compare,

```
> hist(income,
+ breaks=seq(min(income)-1,max(
    income)+50,by=50),
+ probability=TRUE)
> lines(density(income),col="red"
    ,lwd=2)
```



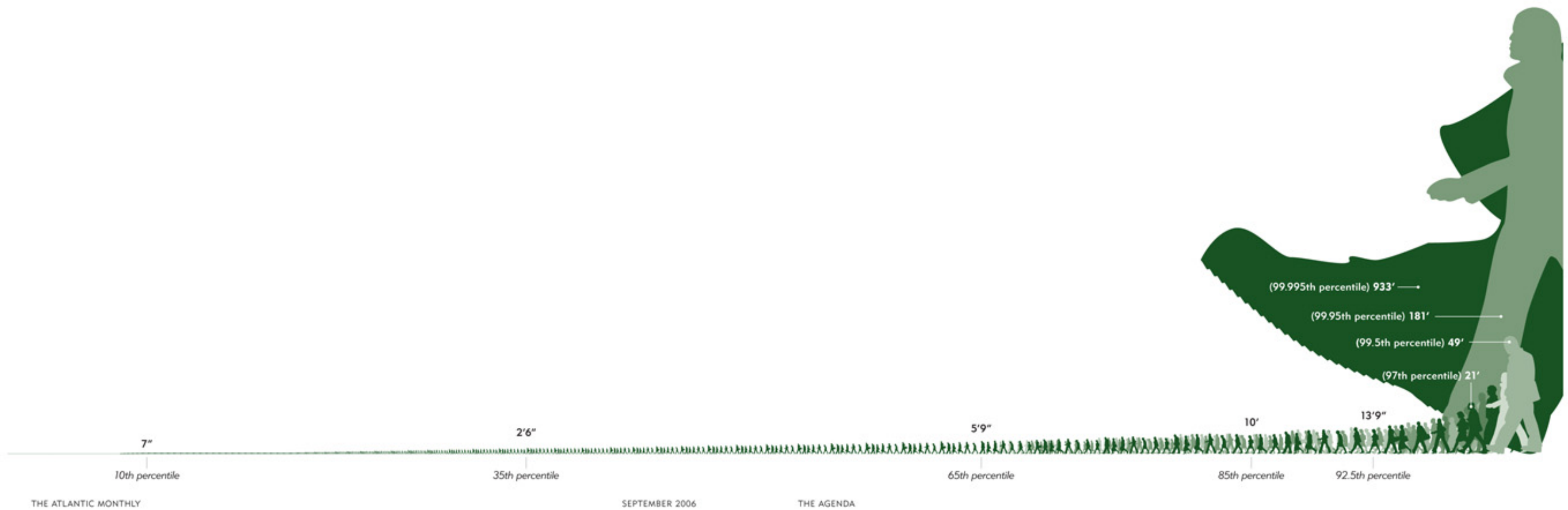
It is more convenient, compare cumulative distribution functions of income, wealth, consumption, grades, etc.

```
> plot(ecdf(income))
```



The Parade of Dwarfs

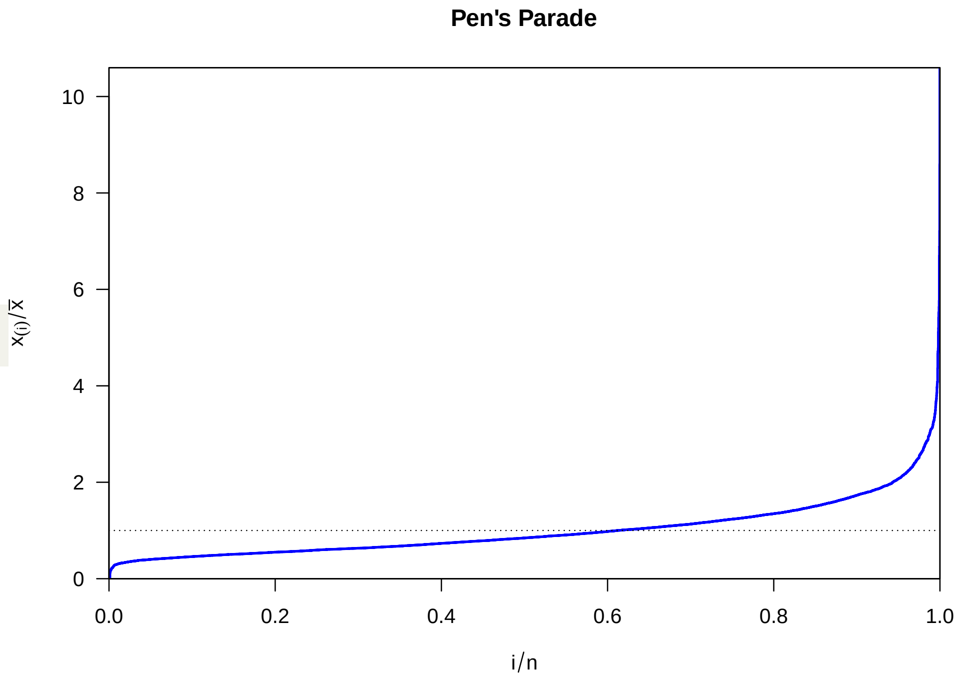
An alternative is to use Pen's parade, also called the parade of dwarfs (and a few giants), "parade van dwergen en een enkele reus".



The height of each person is stretched in the proportion to his or her income everyone is line up in order of height, shortest (poorest) are on the left and tallest (richest) are on the right let them walk some time, like a procession.

c.d.f., quantiles and Lorenz

```
1 > Pen(income)
```



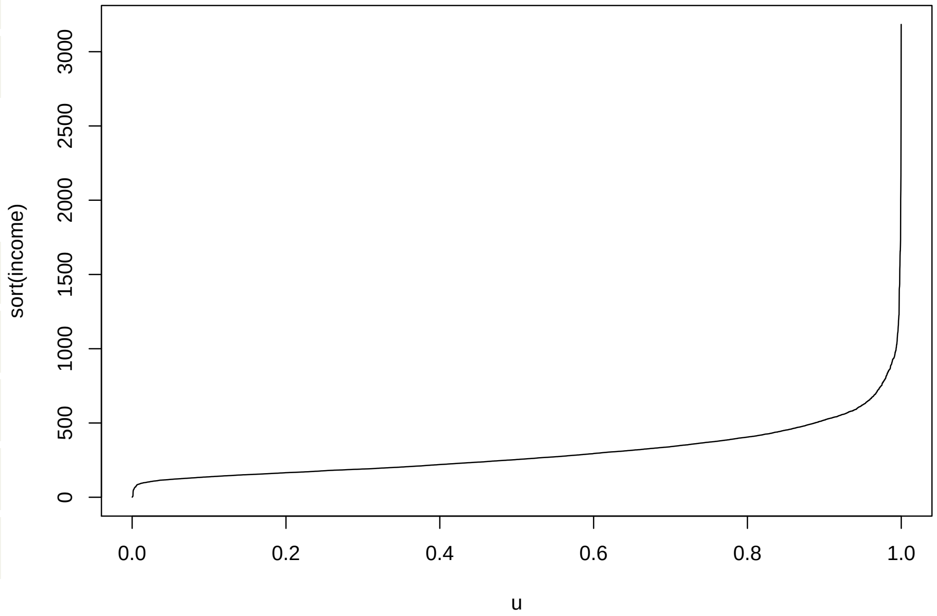
c.d.f., quantiles and Lorenz

This parade of the Dwarfs function is just the quantile function.

```
> q <- function(u) quantile(
    income,u)
```

see also

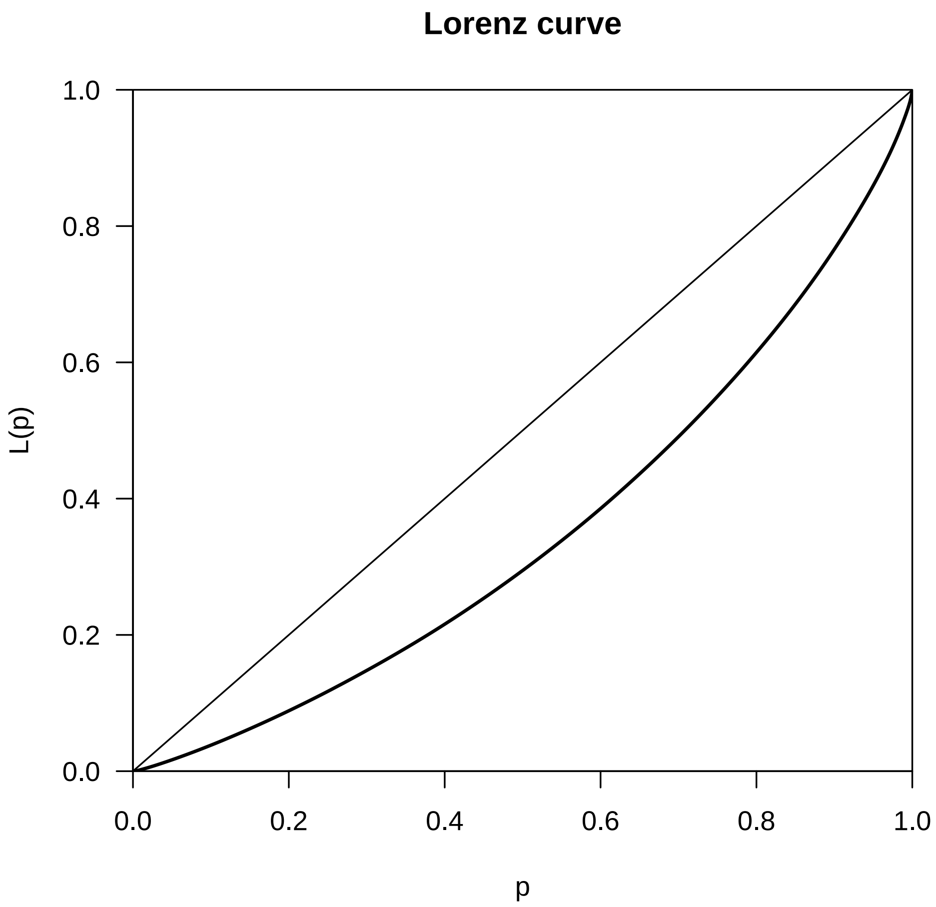
```
> n <- length(income)
> u <- seq(1/(2*n),1-1/(2*n),
    length=n)
> plot(u,sort(income),type="l")
    plot(ecdf(income))
```



c.d.f., quantiles and Lorenz

To get Lorentz curve, we substitute on the y-axis *proportion of incomes* to *incomes*.

```
> library(ineq)
> Lc(income)
> L <- function(u) Lc(income)$L[
    round(u*length(income))]
```



c.d.f., quantiles and Lorenz

	x-axis	y-axis
c.d.f.	income	proportion of population
Pen's parade (quantile)	proportion of population	income
Lorenz curve	proportion of population	proportion of income

Standard statistical measure of dispersion

The variance for a sample $\boldsymbol{X} = \{x_1, \cdots, x_n\}$ is

$$\text{Var}(\boldsymbol{X}) = \frac{1}{n}\sum_{i=1}^{n}[x_i - \overline{x}]^2$$

where the baseline (reference) is $\overline{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$.

```
> var(income)
[1] 34178.43
```

problem it is a quadratic function, $\text{Var}(\alpha\boldsymbol{X}) = \alpha^2\text{Var}(\boldsymbol{X})$.

Standard statistical measure of dispersion

An alternative is the coefficient of variation,

$$\text{cv}(\boldsymbol{X}) = \frac{\sqrt{\text{Var}(\boldsymbol{X})}}{\overline{x}}$$

But not a good measure to capture inequality overall, very sensitive to very high incomes

```
> cv <- function(x) sd(x)/mean(x)
> cv(income)
[1] 0.6154011
```

Standard statistical measure of dispersion

An alternative is to use a logarithmic transformation. Use the logarithmic variance

$$\text{Var}_{\log}(\boldsymbol{X}) = \frac{1}{n}\sum_{i=1}^{n}[\log(x_i) - \log(\overline{x})]^2$$

```
> var_log <- function(x) var(log(x))
> var_log(income)
[1] 0.2921022
```

Those measures are distances on the x-axis.

Standard statistical measure of dispersion

Other inequality measures can be derived from Pen's parade of the Dwarfs, where measures are based on distances on the y-axis, i.e. distances between quantiles.

$$Q_p = F^{-1}(p) \text{ i.e. } F(Q_p) = p$$

e.g. the median is the quantile when $p = 50\%$, the first quartile is the quantile when $p = 25\%$, the first quintile is the quantile when $p = 20\%$, the first decile is the quantile when $p = 10\%$, the first percentile is the quantile when $p = 1\%$

```
> quantile(income,c(.1,.5,.9,.99))
      10%       50%       90%       99%
137.6294  253.9090  519.6887  933.9211
```

Standard statistical measure of dispersion

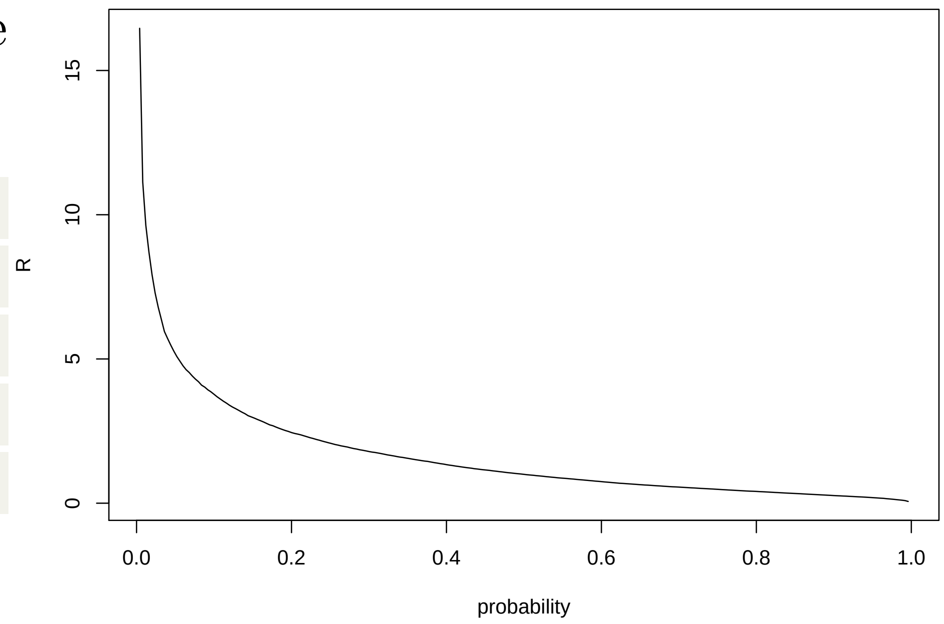
Define the quantile ratio as

$$R_p = \frac{Q_{1-p}}{Q_p}$$

In case of perfect equality, $R_p = 1$.

The most popular one is probably the 90/10 ratio.

```
> R_p <- function(x,p) quantile(x
    ,1-p)/quantile(x,p)
> R_p(income,.1)
  90%
3.776
```



This index measures the gap between the rich and the poor.

E.g. $R_{0.1} = 10$ means that top 10% incomes are more than 10 times higher than the bottom 10% incomes.

Ignores the distribution (apart from the two points), violates transfer principle.

An alternative measure might be Kuznets Ratio, defined from Lorenz curve as the ratio of the share of income earned by the poorest p share of the population and the richest r share of the population,

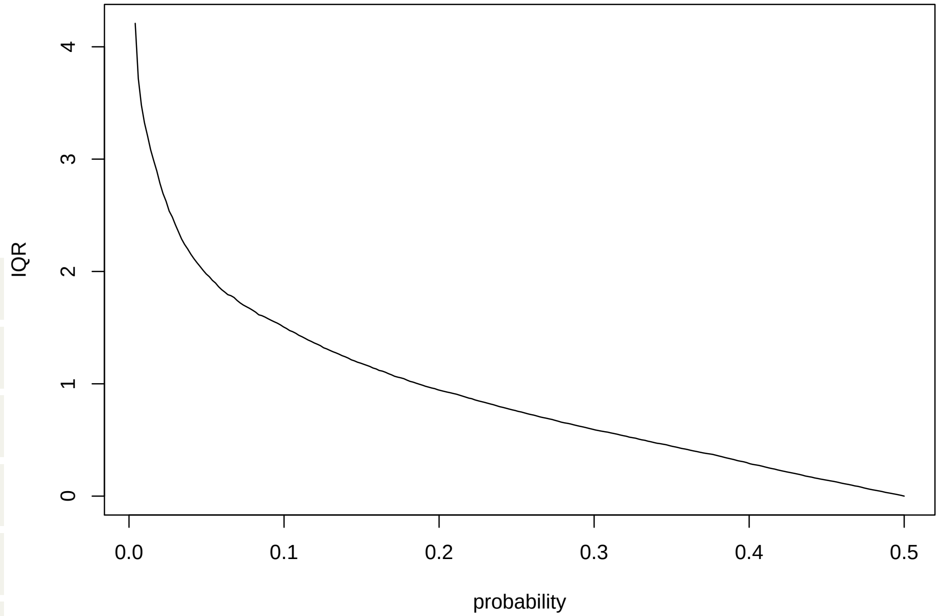
$$I(p,r) = \frac{L(p)}{1 - L(1-r)}$$

But here again, it ignores the distribution between the cutoffs and therefore violates the transfer principle.

An alternative measure can be the IQR, interquantile ratio,

$$IQR_p = \frac{Q_{1-p} - Q_p}{Q_{0.5}}$$

```
> IQR_p <- function(x,p) (
      quantile(x,1-p)-quantile(x,p)
      )/quantile(x,.5)
> IQR_p(income,.1)
       90%
1.504709
```



Problem only focuses on top $(1-p)$-th and bottom p-th proportion. Does not care about what happens between those quantiles.

Standard statistical measure of dispersion

Pen's parade suggest to measure the green area, for some $p \in (0,1)$, M_p,

```
> M_p <- function(x,p){
  a <- seq(0,p,length=251)
  b <- seq(p,1,length=251)
  ya <- quantile(x,p)-quantile(x,a)
  a1 <- sum((ya[1:250]+ya[2:251])/2*p/250)
  yb <- quantile(x,b)-quantile(x,p)
  a2 <- sum((yb[1:250]+yb[2:251])/2*(1-p)/250)
return(a1+a2)}
```

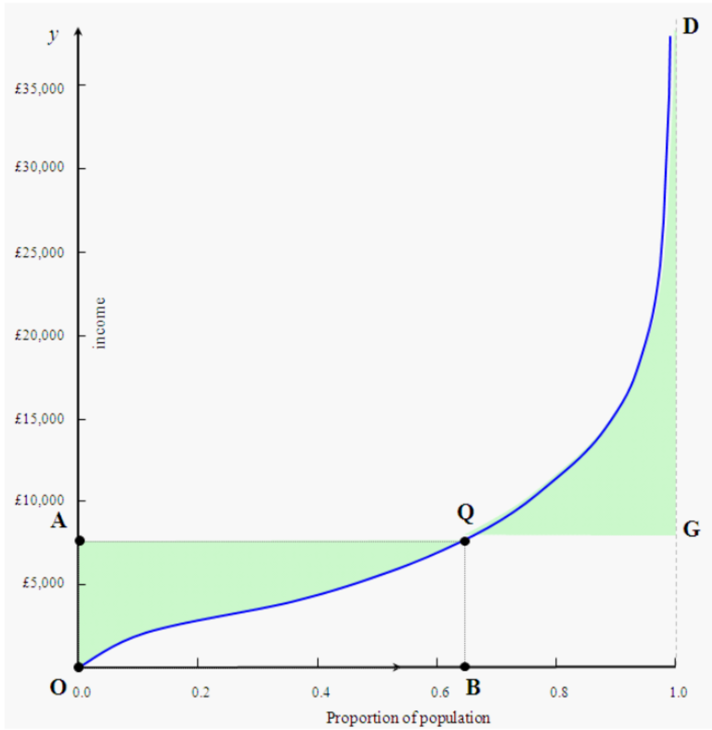


Figure 2.1: The Parade of Dwarfs. UK Income Before Tax, 1984/5. Source: Economic Trends, November 1987

Standard statistical measure of dispersion

Use also the relative mean deviation

$$M(\boldsymbol{X}) = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_i}{\overline{x}} - 1\right|$$

```
> M <- function(x) mean(abs(x/mean(x)-1))
> M(income)
[1] 0.429433
```

in case of perfect equality, $M = 0$

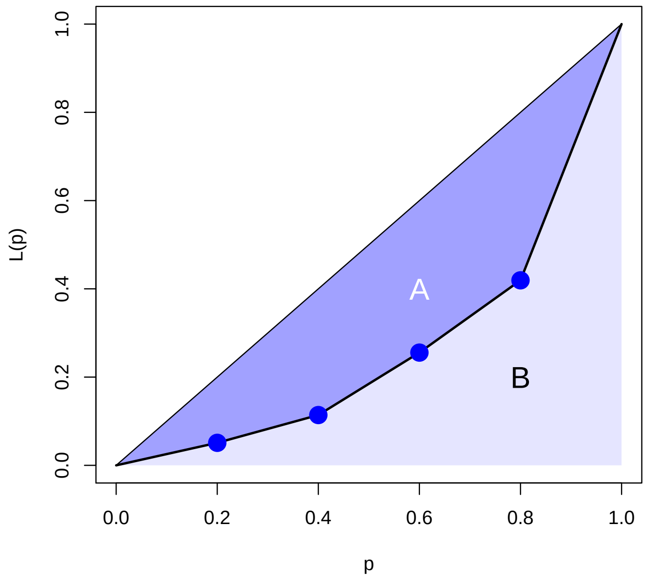
Standard statistical measure of dispersion

Finally, why not use Lorenz curve.

It can be defined using order statistics as

$$G = \frac{2}{n(n-1)\overline{x}} \sum_{i=1}^{n} i \cdot x_{i:n} - \frac{n+1}{n-1}$$

```
> n <- length(income)
> mu <- mean(income)
 2*sum((1:n)*sort(income))/ (mu*n*(n-1))-(n
     +1)/(n-1)
[1] 0.2976282
```



Gini index is defined as the area below the first diagonal and above Lorenz curve

Standard statistical measure of dispersion

$$G(\boldsymbol{X}) = \frac{1}{2n^2\overline{x}} \sum_{i,j=1}^{n} |x_i - x_j|$$

Perfect equality is obtained when $G = 0$.

Remark Gini index can be related to the variance or the coefficient of variation, since

$$\text{Var}(\boldsymbol{X}) = \frac{1}{n}\sum_{i=1}^{n}[x_i - \overline{x}]^2 = \frac{1}{n^2}\sum_{i,j=1}^{n}(x_i - x_j)^2$$

Here,

$$G(\boldsymbol{X}) = \frac{\Delta(\boldsymbol{X})}{2\overline{x}} \text{ with } \Delta(\boldsymbol{X}) = \frac{1}{n^2}\sum_{i,j=1}^{n}|x_i - x_j|$$

```
> ineq(income,"Gini")
[1] 0.2975789
```

Axiomatic Approach for Inequality Indices

Need some rules to say if a principle used to divide a cake of fixed size amongst a fixed number of people is fair, on not.

A standard one is the Anonymity Principle. Let $\boldsymbol{X} = \{x_1, \cdots, x_n\}$, then

$$I(x_1, x_2, \cdots, x_n) = I(x_2, x_1, \cdots, x_n)$$

also called Replication Invariance Principle

The Transfert Principle

for any given income distribution if you take a small amount of income from one person and give it to a richer person then income inequality must increase

Pigou (1912) and Dalton (1920), a transfer from a richer to a poorer person will decrease inequality. Let $\boldsymbol{X} = \{x_1, \cdots, x_n\}$ with $x_1 \leq \cdots \leq x_n$, then

$$I(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) \geqslant I(x_1, \cdots, x_i + \delta, \cdots, x_j - \delta, \cdots, x_n)$$

Nevertheless, not easy to compare, compare e.g. Monday and Tuesday

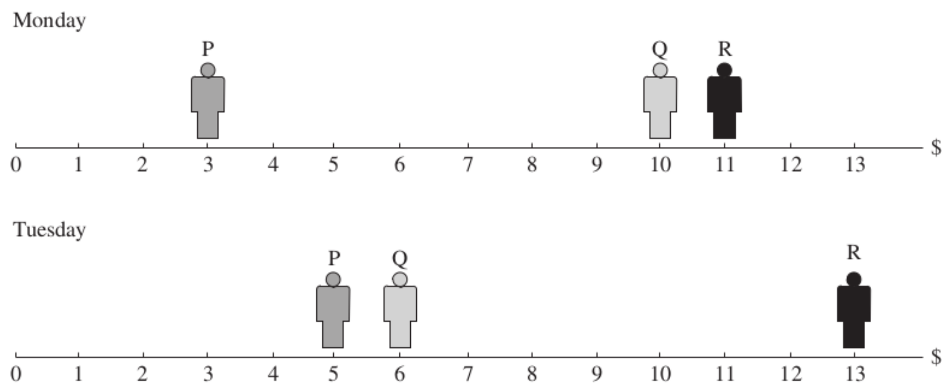


Figure 2.3. Inequality comparisons in a three-person world.

An important concept behind is the idea of mean preserving spread : with those $\pm\delta$ preserve the total wealth.

The Scale Independence Principle

What if double everyone's income ? if standards of living are determined by real income and there is inflation : inequality is unchanged

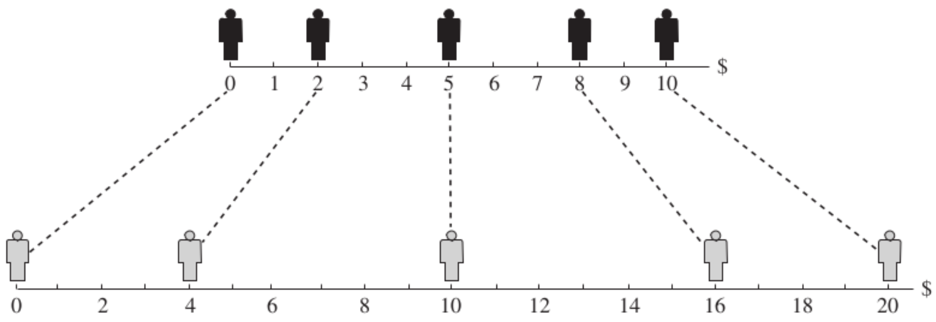


Figure 2.4. Scale independence.

Let $\boldsymbol{X} = \{x_1, \cdots, x_n\}$, then

$$I(\lambda x_1, \cdots, \lambda x_n) = I(x_1, \cdots, x_n)$$

also called Zero-Degree Homogeneity property.

The Population Principle

Consider clones of the economy

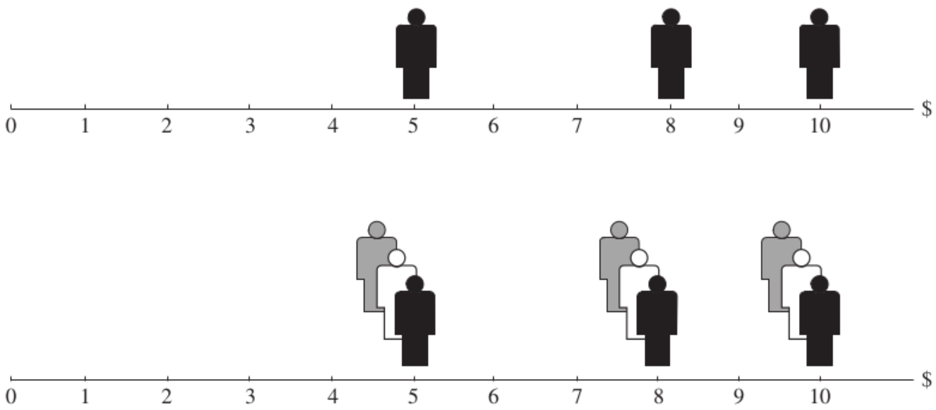


Figure 2.6. The population principle.

$$I(\underbrace{x_1, \cdots, x_1}_{k \text{ times}}, \cdots, \underbrace{x_n, \cdots, x_n}_{k \text{ times}}) = I(x_1, \cdots, x_n)$$

Is it really that simple ?

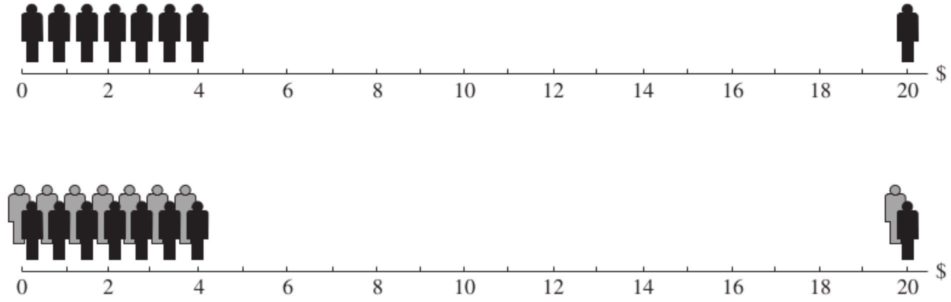


Figure 2.7. Population replication – has inequality fallen?

The Decomposability Principle

Assume that we can decompose inequality by subgroups (based on gender, race, coutries, etc)

According to this principle, if inequality increases in a subgroup, it increases in the whole population, ceteris paribus

$$I(x_1,\cdots,x_n,y_1,\cdots,y_n)\leq I(x_1^\star,\cdots,x_n^\star,y_1,\cdots,y_n)$$

as long as $I(x_1,\cdots,x_n)\leq I(x_1^\star,\cdots,x_n^\star)$.

Consider two groups, $\boldsymbol{X}$ and $\boldsymbol{X}^{\star}$

Then add the same subgroup $\boldsymbol{Y}$ to both $\boldsymbol{X}$ and $\boldsymbol{X}^{\star}$

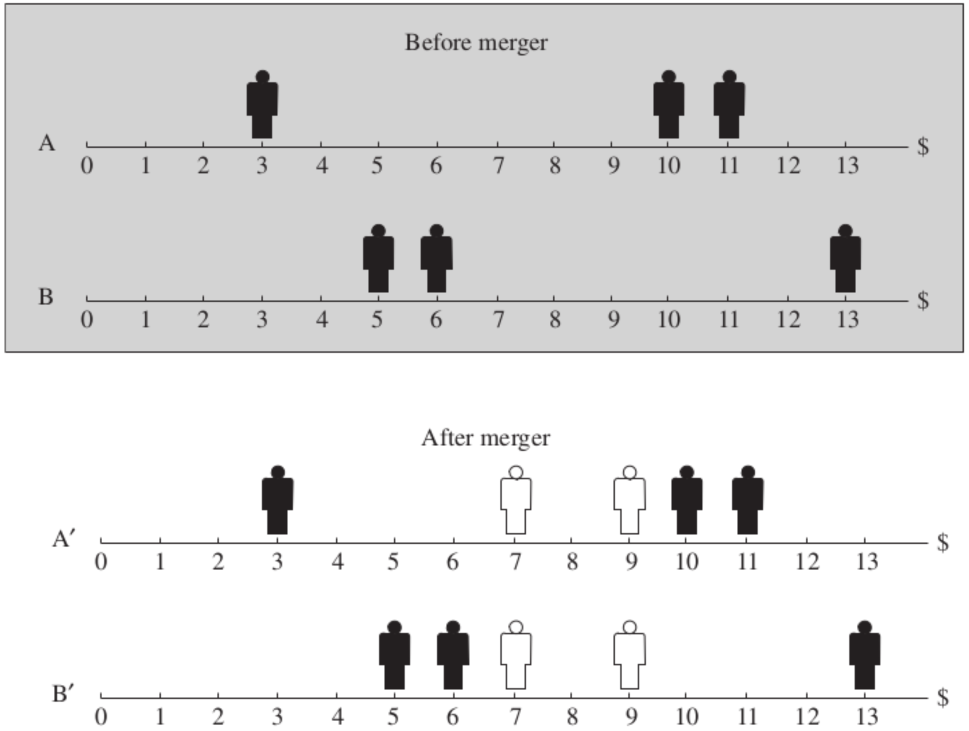


Figure 2.8. Decomposability.

Axiomatic Approach for Inequality Indices

Any inequality measure that simultaneously satisfies the properties of the principle of transfers, scale independence, population principle and decomposability must be expressible in the form

$$E_\xi = \frac{1}{\xi^2 - \xi}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{\overline{x}}\right)^{\xi} - 1\right)$$

for some $\xi \in \mathbb{R}$. This is the generalized entropy measure.

```
> entropy(income,0)
[1] 0.1456604
> entropy(income,.5)
[1] 0.1446105
> entropy(income,1)
[1] 0.1506973
> entropy(income,2)
[1] 0.1893279
```

The higher ξ, the more sensitive to high incomes.

Remark rule of thumb, take $\xi \in [-1, +2]$.

When $\xi = 0$, the mean logarithmic deviation (MLD),

$$MLD = E_0 = -\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{x_i}{\overline{x}}\right)$$

When $\xi = 1$, the Theil index

$$T = E_1 = \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{\overline{x}} \log\left(\frac{x_i}{\overline{x}}\right)$$

```
> Theil(income)
[1] 0.1506973
```

When $\xi = 2$, the index can be related to the coefficient of variation

$$E_2 = \frac{[\text{coefficient of variation}]^2}{2}$$

In a 3-person economy, it is possible to visualize curve of iso-indices,

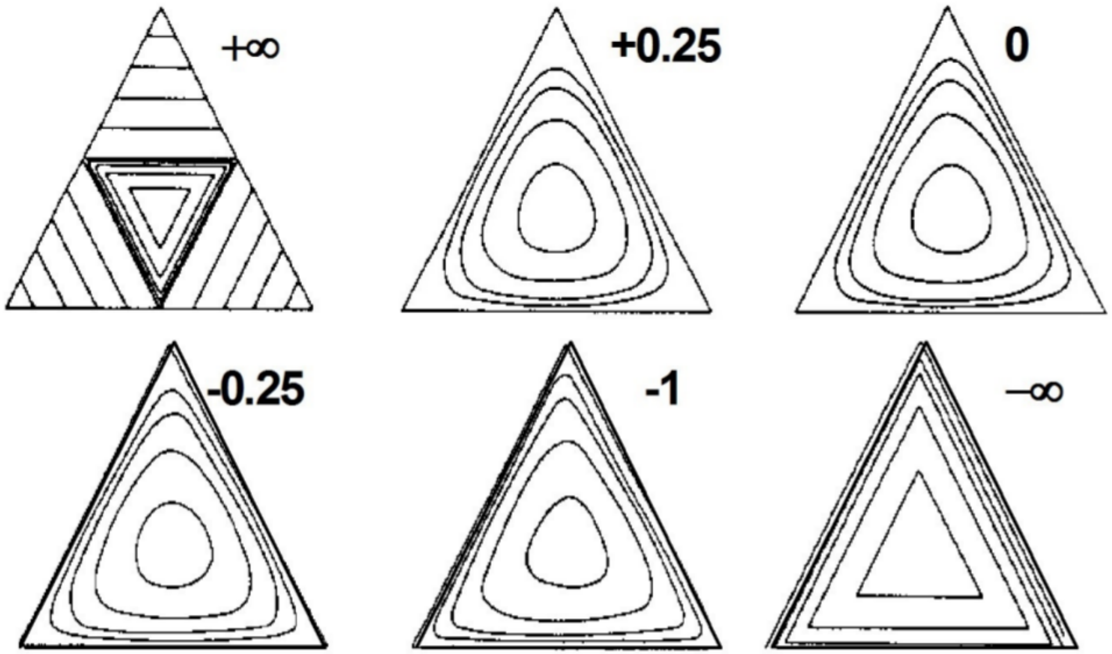


Figure 7: Generalised entropy contours for different values of α

A related index is Atkinson inequality index,

$$A_\epsilon = 1 - \left(\frac{1}{n} \sum_{i=1}^{n} \left(\frac{x_i}{\overline{x}} \right)^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}}$$

with $\epsilon \geq 0$.

```
> Atkinson(income,0.5)
[1] 0.07099824
> Atkinson(income,1)
[1] 0.1355487
```

In the case where $\varepsilon \to 1$, we obtain

$$A_1 = 1 - \prod_{i=1}^{n} \left(\frac{x_i}{\overline{x}}\right)^{\frac{1}{n}}$$

ϵ is usually interpreted as an aversion to inequality index.

Observe that

$$A_\epsilon = 1 - [(\epsilon^2 - \epsilon)E_{1-\epsilon} + 1]^{\frac{1}{1-\epsilon}}$$

and the limiting case $A_1 = 1 - \exp[-E_0]$.

Thus, the Atkinson index is ordinally equivalent to the GE index, since they produce the same ranking of different distributions.

Consider indices obtained when $\boldsymbol{X}$ is obtained from a $LN(0, \sigma^2)$ distribution and from a $\mathcal{P}(\alpha)$ distribution.

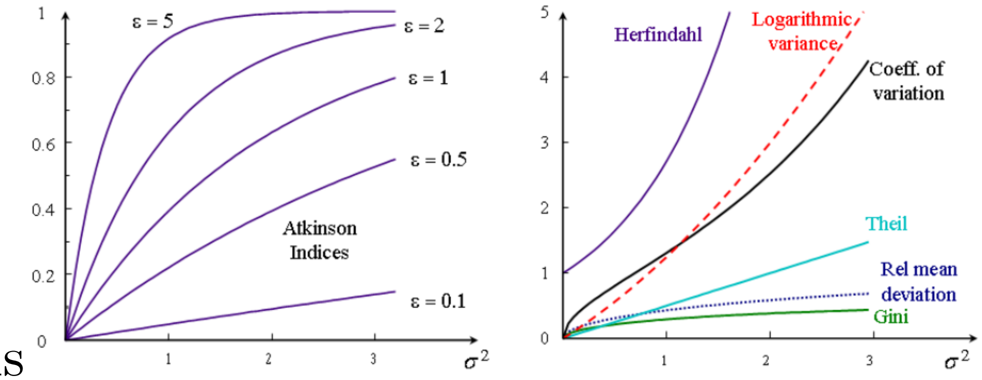


Figure 4.4: Inequality and the Lognormal parameter σ^2

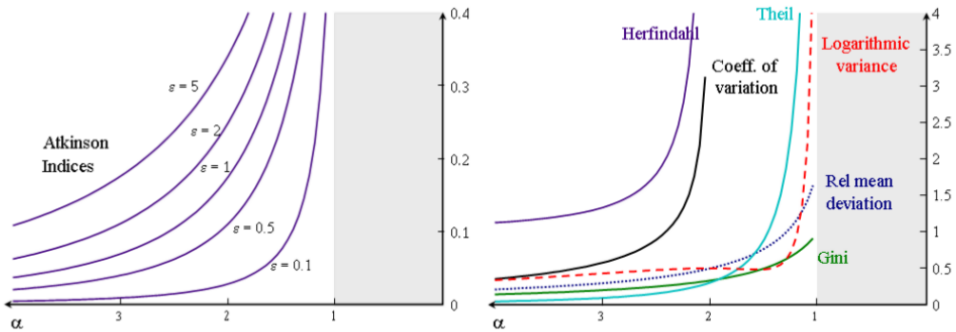


Figure 4.9: Inequality and Pareto's α

Changing the Axioms

Is there an agreement about the axioms ?

For instance, no unanimous agreement on the scale independence axiom,

Why not a translation independence axiom ?

Translation Independence Principle : if every incomes are increased by the same amount, the inequality measure is unchanged

Given $\boldsymbol{X} = (x_1, \cdots, x_n)$,

$$I(x_1, \cdots, x_n) = I(x_1 + h, \cdots, x_n + h)$$

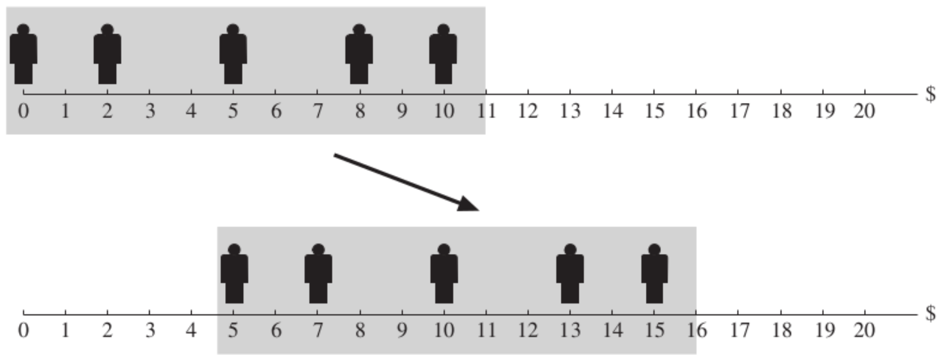


Figure 2.5. Translation independence.

If we change the scale independence principle by this translation independence, we get other indices.

Changing the Axioms

Kolm indices satisfy the principle of transfers, translation independence, population principle and decomposability

$$K_\theta = \log\left(\frac{1}{n}\sum_{i=1}^{n} e^{\theta[x_i-\overline{x}]}\right)$$

```
> Kolm(income,1)
[1] 291.5878
> Kolm(income,.5)
[1] 283.9989
```

From Measuring to Ordering

Over time, between countries, before/after tax, etc.

$\boldsymbol{X}$ is said to be Lorenz-dominated by $\boldsymbol{Y}$ if $L_{\boldsymbol{X}} \leq L_{\boldsymbol{Y}}$. In that case $\boldsymbol{Y}$ is more equal, or less inequal.

In such a case, $\boldsymbol{X}$ can be reached from $\boldsymbol{Y}$ by a sequence of poorer-to-richer pairwiser income transfers.

In that case, any inequality measure satisfying the population principle, scale independence, anonymity and principle of transfers axioms are consistent with the Lorenz dominance (namely Theil, Gini, MLD, Generalized Entropy and Atkinson).

Remark A regressive transfer will move the Lorenz curve further away from the diagonal. So satisfies transfer principle. And it satisfies also the scale invariance property.

Example if $X_i \sim \mathcal{P}(\alpha_i, x_i)$,

$$L_{\boldsymbol{X}_1} \leq L_{\boldsymbol{X}_2} \longleftrightarrow \alpha_1 \leq \alpha_2$$

and if $X_i \sim LN(\mu_i, \sigma_i^2)$,

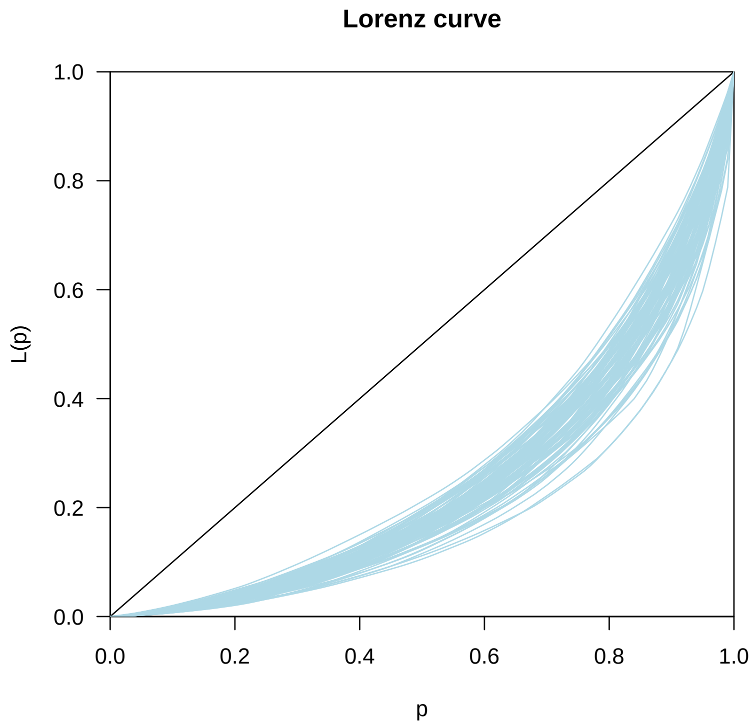
$$L_{\boldsymbol{X}_1} \leq L_{\boldsymbol{X}_2} \longleftrightarrow \sigma_1^2 \geq \sigma_2^2$$

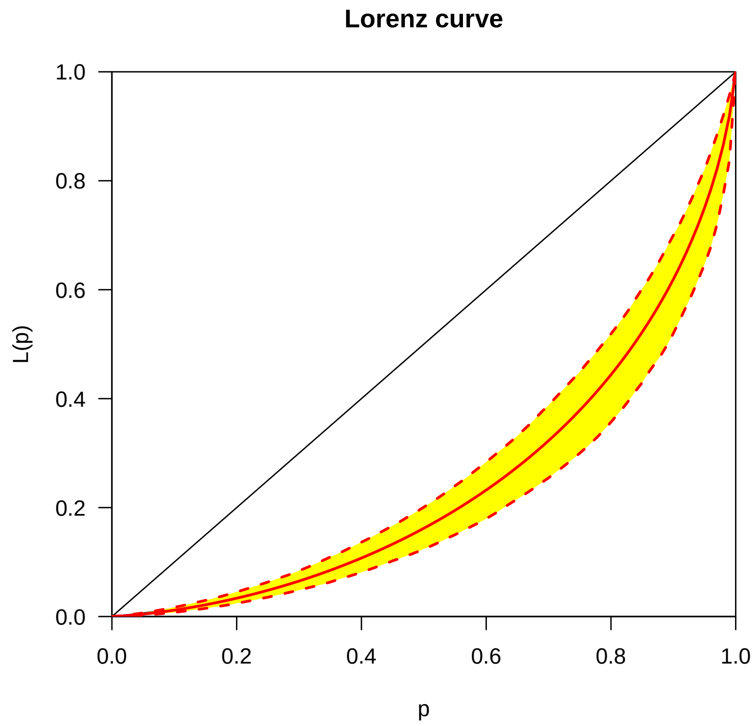
Lorenz dominance is a relation that is incomplete : when Lorenz curves cross, the criterion cannot decide between the two distributions.

$\rightarrow$ the ranking is considered unambiguous.

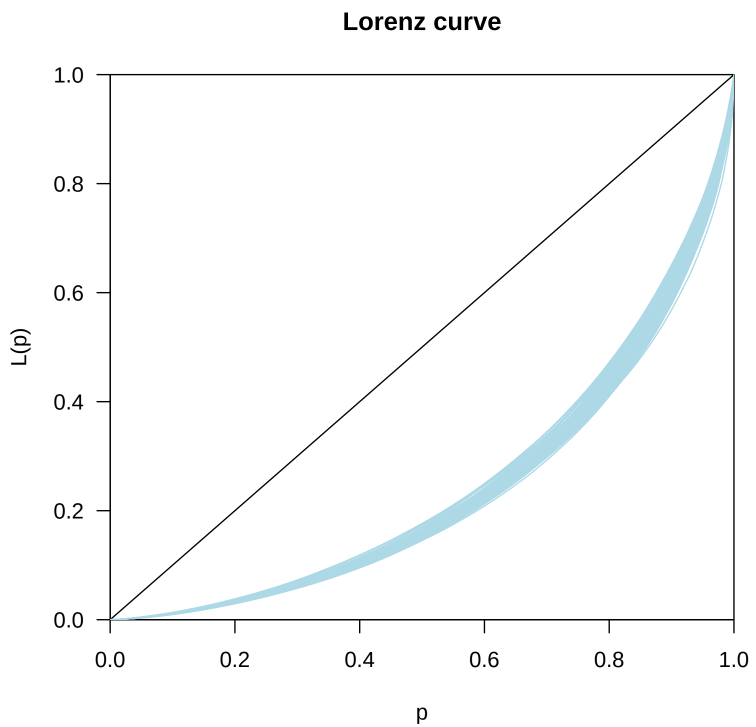
Further, one should take into account possible random noise.

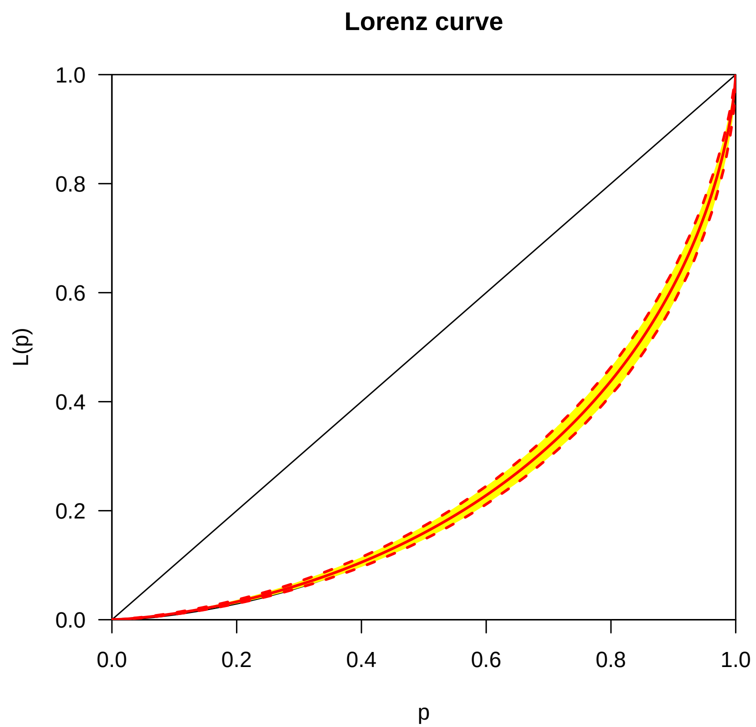
Consider some sample $\{x_1, \cdots, x_n\}$ from a $LN(0,1)$ distribution, with $n = 100$. The 95% confidence interval is





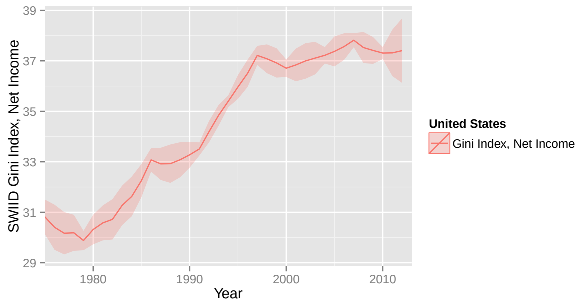
Consider some sample $\{x_1, \cdots, x_n\}$ from a $LN(0,1)$ distribution, with $n = 1,000$. The 95% confidence interval is


Lorenz curve
L(p)
p
0.0
0.2
0.4
0.6
0.8
1.0

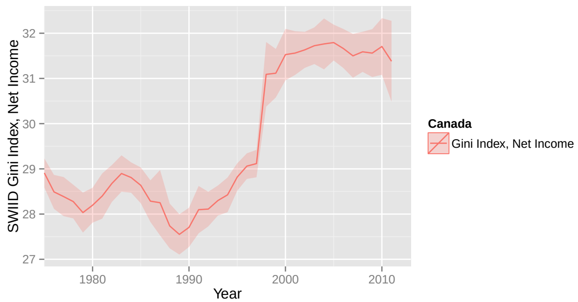

Lorenz curve
L(p)
p
0.0
0.2
0.4
0.6
0.8
1.0

Looking for Confidence

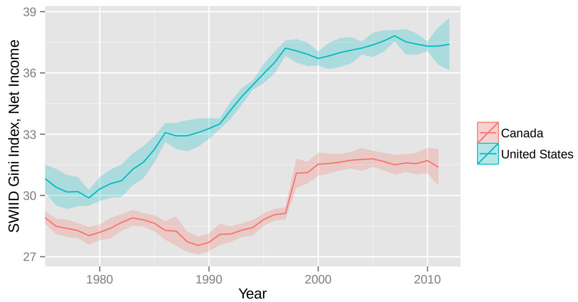
See e.g. http://myweb.uiowa.edu/fsolt/swiid/, for the estimation of Gini index over time + over several countries.



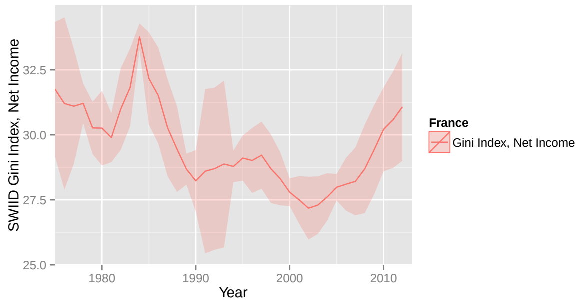
Note: Solid lines indicate mean estimates; shaded regions indicate the associated 95% confidence intervals.
Source: Standardized World Income Inequality Database v5.0 (Solt 2014).



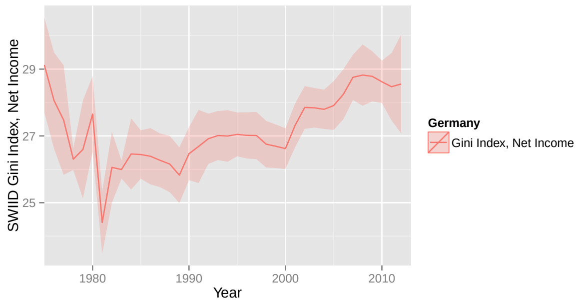
Note: Solid lines indicate mean estimates; shaded regions indicate the associated 95% confidence intervals.
Source: Standardized World Income Inequality Database v5.0 (Solt 2014).



Note: Solid lines indicate mean estimates; shaded regions indicate the associated 95% confidence intervals.
Source: Standardized World Income Inequality Database v5.0 (Solt 2014).



Note: Solid lines indicate mean estimates; shaded regions indicate the associated 95% confidence intervals.
Source: Standardized World Income Inequality Database v5.0 (Solt 2014).



Note: Solid lines indicate mean estimates; shaded regions indicate the associated 95% confidence intervals.
Source: Standardized World Income Inequality Database v5.0 (Solt 2014).

Note: Solid lines indicate mean estimates; shaded regions indicate the associated 95% confidence intervals.
Source: Standardized World Income Inequality Database v5.0 (Solt 2014).

Looking for Confidence

To get confidence interval for indices, use bootsrap techniques (see last week). The code is simply

```
> IC <- function(x,f,n=1000,alpha=.95){
+ F=rep(NA,n)
+ for(i in 1:n){
+ F[i]=f(sample(x,size=length(x),replace=TRUE))}
+ return(quantile(F,c((1-alpha)/2,1-(1-alpha)/2)))}
```

For instance,

```
> IC(income,Gini)
      2.5%     97.5%
0.2915897 0.3039454
```

(the sample is rather large, $n = 6,043$.

Looking for Confidence

```
> IC(income,Gini)
     2.5%     97.5%
0.2915897 0.3039454
> IC(income,Theil)
     2.5%     97.5%
0.1421775 0.1595012
> IC(income,entropy)
     2.5%     97.5%
0.1377267 0.1517201
```

Back on Gini Index

We've seen Gini index as an area,

$$G = 2\int_0^1 [p - L(p)]dp = 1 - 2\int_0^1 L(p)dp$$

Using integration by parts, $u' = 1$ and $v = L(p)$,

$$G = -1 + 2\int_0^1 pL'(p)dp = \frac{2}{\mu}\left(\int_0^\infty yF(y)f(y)dy - \frac{\mu}{2}\right)$$

using a change of variables, $p = F(y)$ and because $L'(p) = F^{-1}(p)/\mu = y/mu$.

Thus

$$G = \frac{2}{\mu}\text{cov}(y, F(y))$$

$\rightarrow$ Gini index is proportional to the covariance between the income and its rank.

Back on Gini Index

Using integration be parts, one can then write

$$G = \frac{1}{2}\int_0^\infty F(x)[1-F(x)]dx = 1 - \frac{1}{\mu}\int_0^\infty [1-F(x)]^2 dx.$$

which can also be writen

$$G = \frac{1}{2\mu}\int_{\mathbb{R}_+^2} |x-y| dF(x)dF(y)$$

(see previous discussion on connexions between Gini index and the variance)

Decomposition(s)

When studying inequalities, it might be interesting to discussion possible decompostions either by subgroups, or by sources,

— subgroups decomposition, e.g Male/Female, Rural/Urban see FAO (2006, fao.org)

— source decomposition, e.g earnings/gvnt benefits/investment/pension, etc, see slide 41 #1 and FAO (2006, fao.org)

For the variance, decomposition per groups is related to ANOVA,

$$\text{Var}(Y) = \underbrace{\mathbb{E}[\text{Var}(Y|X)]}_{\text{within}} + \underbrace{\text{Var}(\mathbb{E}[Y|X])}_{\text{between}}$$

Hence, if $X \in \{x_1, \cdots, x_k\}$ (k subgroups),

$$\text{Var}(Y) = \underbrace{\sum_k p_k \text{Var}(Y| \text{ group } k)}_{\text{within}} + \underbrace{\text{Var}(\mathbb{E}[Y|X])}_{\text{between}}$$

Decomposition(s)

For Gini index, it is possible to write

$$\text{G}(Y) = \underbrace{\sum_k \omega_k \text{G}(Y|\text{ group } k)}_{\text{within}} + \underbrace{\text{G}(\overline{Y})}_{\text{between}} + \text{residual}$$

for some weights $\boldsymbol{\omega}$, where the between term is the Gini index between subgroup means. But the decomposition is not perfect.

More generally, for General Entropy indices,

$$\text{E}_\xi(Y) = \underbrace{\sum_k \omega_k \text{E}_\xi(Y|\text{ group } k)}_{\text{within}} + \underbrace{\text{E}_\xi(\overline{Y})}_{\text{between}}$$

where $\text{E}_\xi(\overline{Y})$ is the entropy on the subgroup means

$$\omega_k = \left(\frac{\overline{Y}_k}{\overline{Y}}\right)^\xi (p_k)^{1-\xi}$$

Decomposition(s)

Now, a decomposition per source, i.e. $Y_i = Y_{1,i} + \cdots + Y_{k,i} + \cdots$, among sources.

For Gini index natural decomposition was suggested by Lerman & Yitzhaki (1985, jstor.org)

$$G(Y) = \frac{2}{\overline{\overline{Y}}}\text{cov}(Y, F(Y)) = \sum_k \underbrace{\frac{2}{\overline{\overline{Y}}}\text{cov}(Y_{k}, F(Y))}_{k\text{-th contribution}}$$

thus, it is based on the covariance between the k-th source and the ranks based on cumulated incomes.

Similarly for Theil index,

$$T(Y) = \sum_k \underbrace{\frac{1}{n}\sum_i \left(\frac{Y_{k,i}}{\overline{\overline{Y}}}\right)\log\left(\frac{Y_i}{\overline{\overline{Y}}}\right)}_{k\text{-th contribution}}$$

Decomposition(s)

It is possible to use Shapley value for decomposition of indices $\mathcal{I}(\cdot)$. Consider m groups, $N=\{1,\cdots,m\}$, and definie $\mathcal{I}(S)=I(\boldsymbol{x}_S)$ where $S\subset N$. Then Shapley value yields

$$\phi_k(v)=\sum_{S\subseteq N\backslash\{k\}}\frac{|S|!\,(m-|S|-1)!}{m!}(\mathcal{I}(S\cup\{k\})-\mathcal{I}(S))$$

Regression ?

Galton (1870, galton.org, 1886, galton.org) and Pearson & Lee (1896, jstor.org, 1903 jstor.org) studied genetic transmission of characterisitcs, e.g. the heigth.

On average the child of tall parents is taller than other children, but less than his parents.

"I have called this peculiarity by the name of regression', Francis Galton, 1886.

Regression *towards* Mediocrity *in* Hereditary Stature.
By Francis Galton, F.R.S., &c.

Table 8.1. Galton's 1885 cross-tabulation of 928 adult children born of 205 midparents, by their height and their midparent's height.

Height of the mid-parent in inches	Height of the adult child <61.7	62.2	63.2	64.2	65.2	66.2	67.2	68.2	69.2	70.2	71.2	72.2	73.2	>73.7	Total no. of adult children	Total no. of mid-parents	Medians
> 73.0	—	—	—	—	—	—	—	—	—	—	—	1	3	—	4	5	—
72.5	—	—	—	—	—	—	—	1	2	1	2	7	2	4	19	6	72.2
71.5	—	—	—	—	1	3	4	3	5	10	4	9	2	2	43	11	69.9
70.5	1	—	1	—	1	1	3	12	18	14	7	4	3	3	68	22	69.5
69.5	—	—	1	16	4	17	27	20	33	25	20	11	4	5	183	41	68.9
68.5	1	—	7	11	16	25	31	34	48	21	18	4	3	—	219	49	68.2
67.5	—	3	5	14	15	36	38	28	38	19	11	4	—	—	211	33	67.6
66.5	—	3	3	5	2	17	17	14	13	4	—	—	—	—	78	20	67.2
65.5	1	—	9	5	7	11	11	7	7	5	2	1	—	—	66	12	66.7
64.5	1	1	4	4	1	5	5	—	2	—	—	—	—	—	23	5	65.8
< 64.0	1	—	2	4	1	2	2	1	1	—	—	—	—	—	14	1	—
Totals	5	7	32	59	48	117	138	120	167	99	64	41	17	14	928	205	—
Medians	—	—	66.3	67.8	67.9	67.7	67.9	68.3	68.5	69.0	69.0	70.0	—	—	—	—	—

Source: Galton (1886a).

Note: All female heights were multiplied by 1.08 before tabulation. Galton added an explanatory footnote to the table: "In calculating the Medians, the entries have been taken as referring to the middle of the squares in which they stand. The reason why the headings run 62.2, 63.2, &c., instead of 62.5, 63.5, &c., is that the observations are unequally distributed between 62 and 63, 63 and 64, &c., there being a strong bias in favour of integral inches. After careful consideration, I concluded that the headings, as adopted, best satisfied the conditions. This inequality was not apparent in the case of the Mid-parents." Galton republished these data in 1889, where they are referred to as the R.F.F. Data (Record of Family Faculties); he then noted that the first row must be in error (four children cannot have five sets of parents), but he claimed that "the bottom line, which looks suspicious, is correct" (p. 208).

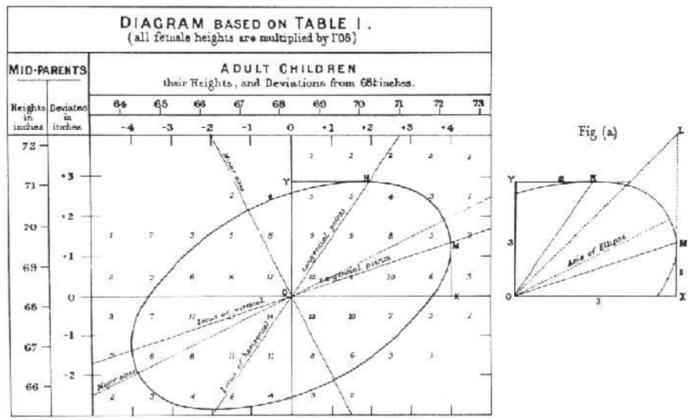
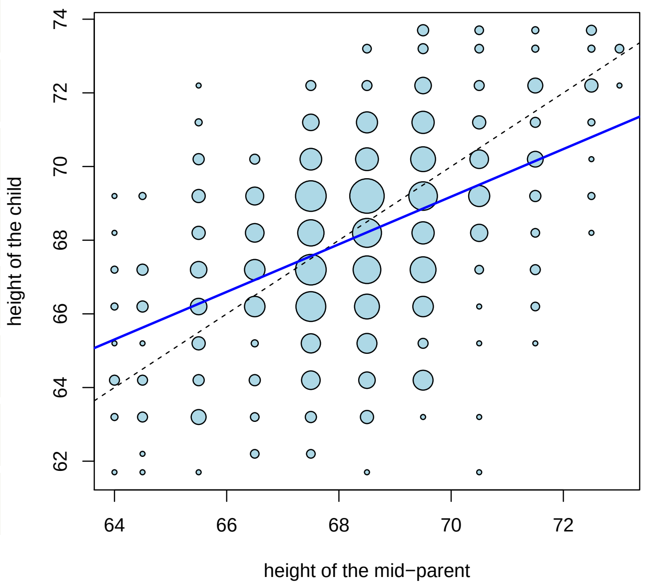


Figure 8.7. Galton's smoothed rendition of Table 8.1, with one of the "concentric and similar ellipses" drawn in. The geometric relationship of the two regression lines to the ellipse is also shown. (From Galton, 1886a.)

Regression ?

```
> library(HistData)
> attach(Galton)
> Galton$count <- 1
> df <- aggregate(Galton, by=list(parent,
    child), FUN=sum)[,c(1,2,5)]
> plot(df[,1:2],cex=sqrt(df[,3]/3))
> abline(a=0,b=1,lty=2)
> abline(lm(child~parent,data=Galton))
```



Least Squares ?

Recall that

$$\begin{cases} \mathbb{E}(Y) = \underset{m\in\mathbb{R}}{\text{argmin}} \left\{\|Y-m\|_{\ell_2}^2 = \mathbb{E}\left([Y-m]^2\right)\right\} \\ \text{Var}(Y) = \underset{m\in\mathbb{R}}{\min} \left\{\mathbb{E}\left([Y-m]^2\right)\right\} = \mathbb{E}\left([Y-\mathbb{E}(Y)]^2\right) \end{cases}$$

The empirical version is

$$\begin{cases} \overline{y} = \underset{m\in\mathbb{R}}{\text{argmin}} \left\{\sum_{i=1}^n \frac{1}{n}[y_i-m]^2\right\} \\ s^2 = \underset{m\in\mathbb{R}}{\min} \left\{\sum_{i=1}^n \frac{1}{n}[y_i-m]^2\right\} = \sum_{i=1}^n \frac{1}{n}[y_i-\overline{y}]^2 \end{cases}$$

The conditional version is

$$\begin{cases} \mathbb{E}(Y|\boldsymbol{X}) = \underset{\varphi:\mathbb{R}^k\to\mathbb{R}}{\text{argmin}} \left\{\|Y-\varphi(\boldsymbol{X})\|_{\ell_2}^2 = \mathbb{E}\left([Y-\varphi(\boldsymbol{X})]^2\right)\right\} \\ \text{Var}(Y|\boldsymbol{X}) = \underset{\varphi:\mathbb{R}^k\to\mathbb{R}}{\min} \left\{\mathbb{E}\left([Y-\varphi(\boldsymbol{X})]^2\right)\right\} = \mathbb{E}\left([Y-\mathbb{E}(Y|\boldsymbol{X})]^2\right) \end{cases}$$

Changing the Distance in Least-Squares ?

One might consider $\widehat{\boldsymbol{\beta}} \in \text{argmin}\left\{\sum_{i=1}^{n} |Y_i - \boldsymbol{X}_i^\mathsf{T}\boldsymbol{\beta}|\right\}$, based on the ℓ_1-norm, and not the ℓ_2-norm.

This is the least-absolute deviation estimator, related to the median regression, since $\text{median}(X) = \text{argmin}\{\mathbb{E}|X - x|\}$.

More generally, assume that, for some function $R(\cdot)$,

$$\widehat{\boldsymbol{\beta}} \in \text{argmin}\left\{\sum_{i=1}^{n} R(Y_i - \boldsymbol{X}_i^\mathsf{T}\boldsymbol{\beta})\right\}$$

If R is differentiable, the first order condition would be

$$\sum_{i=1}^{n} R'\left(Y_i - \boldsymbol{X}_i^\mathsf{T}\boldsymbol{\beta}\right) \cdot \boldsymbol{X}_i^\mathsf{T} = 0.$$

Changing the Distance in Least-Squares ?

i.e.

$$\sum_{i=1}^{n} \underbrace{\omega\left(Y_i - \boldsymbol{X}_i^\top\boldsymbol{\beta}\right)}_{\omega_i} \cdot \left(Y_i - \boldsymbol{X}_i^\top\boldsymbol{\beta}\right) \boldsymbol{X}_i^\top = 0 \text{ with } \omega(x) = \frac{R'(x)}{x},$$

It is the first order condition of a weighted ℓ_2 regression.

To obtain the ℓ_1-regression, observe that $\omega = |\varepsilon|^{-1}$

Changing the Distance in Least-Squares ?

$\Longrightarrow$ use iterative (weighted) least-square regressions.

Start with some standard ℓ_2 regression

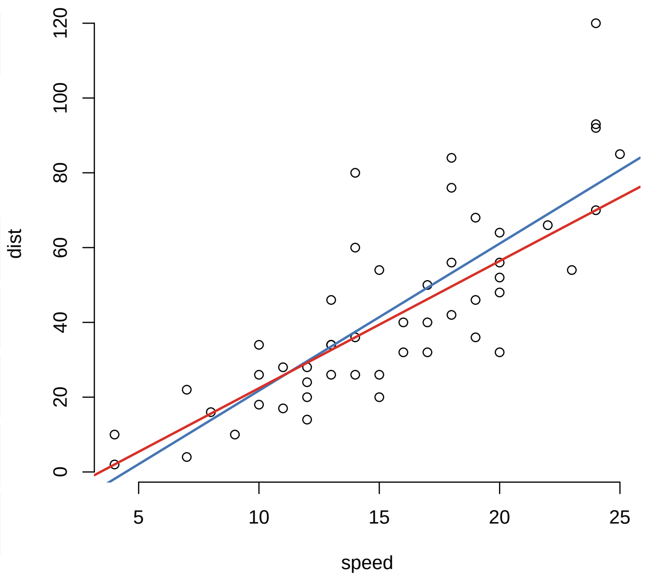
```
> reg_0 <- lm(Y~X, data=db)
```

For the ℓ_1 regression consider weight function

```
> omega <- function(e) 1/abs(e)
```

Then consider the following iterative algorithm

```
> resid <- residuals(reg_0)
> for(i in 1:100){
+ W <- omega(e)
+ reg <- lm(Y~X, data=db, weights=W)
+ e <- residualts(reg)}
```



Quantile Regression

Observe that, for all $\tau \in (0,1)$

$$Q_X(\tau) = F_X^{-1}(\tau) = \underset{m\in\mathbb{R}}{\text{argmin}}\,\{\mathbb{E}[R_\tau(X-m)]\}$$

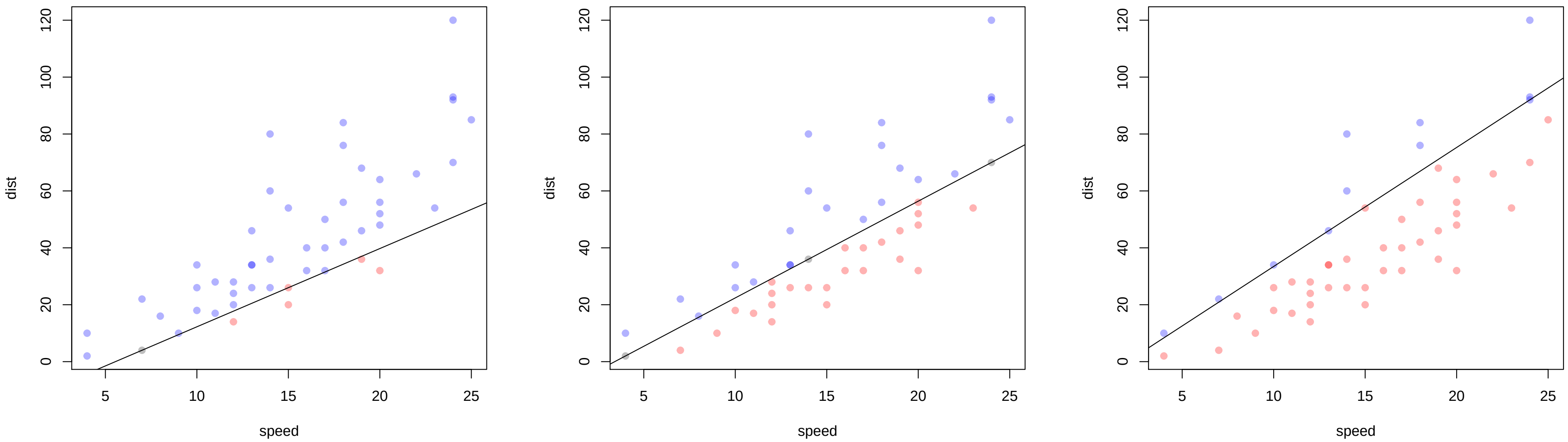
where $R_\tau(x) = [\tau - \mathbf{1}(x<0)]\cdot x$.

From a statistical point of view

$$\widehat{Q}_{\boldsymbol{x}}(\tau) = \underset{m\in\mathbb{R}}{\text{argmin}}\left\{\frac{1}{n}\sum_{i=1}^{n} R_\tau(x_i - m)\right\}.$$

The quantile-τ regression

$$\widehat{\boldsymbol{\beta}} = \text{argmin}\left\{\sum_{i=1}^{n}\mathcal{R}_\tau(Y_i - \boldsymbol{X}_i^\mathsf{T}\boldsymbol{\beta})\right\}.$$



There are $n(1-p)$ points in the upper region, and np in the lower one.

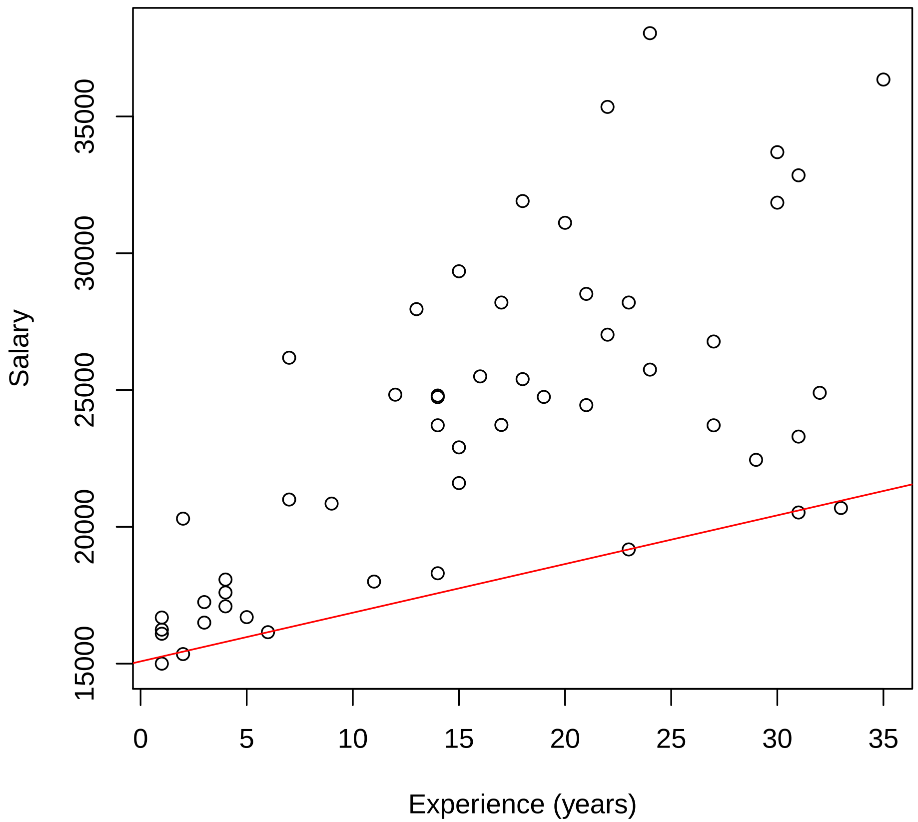
```
> library(quantreg)
> fit1 <- rq(y ~ x1 + x2, tau = .1, data = df)
```

see cran.r-project.org.

Quantile Regression : Empirical Analysis

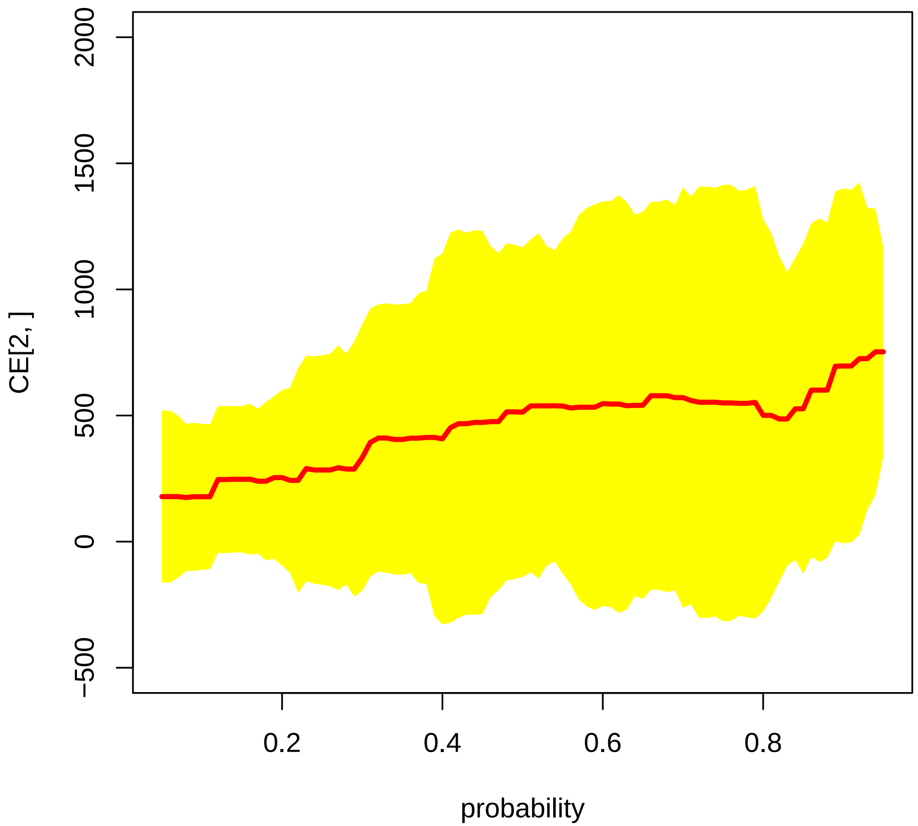
Consider here some salaries, as a function of the experience (in years), see data.princeton.edu

```
> salary=read.table("http://data.
    princeton.edu/wws509/datasets
    /salary.dat",header=TRUE)
> library(quantreg)
> plot(salary$yd,c)
> abline(rq(sl~yd,tau=.1,data=
    salary),col="red")
```



Quantile Regression : Empirical Analysis

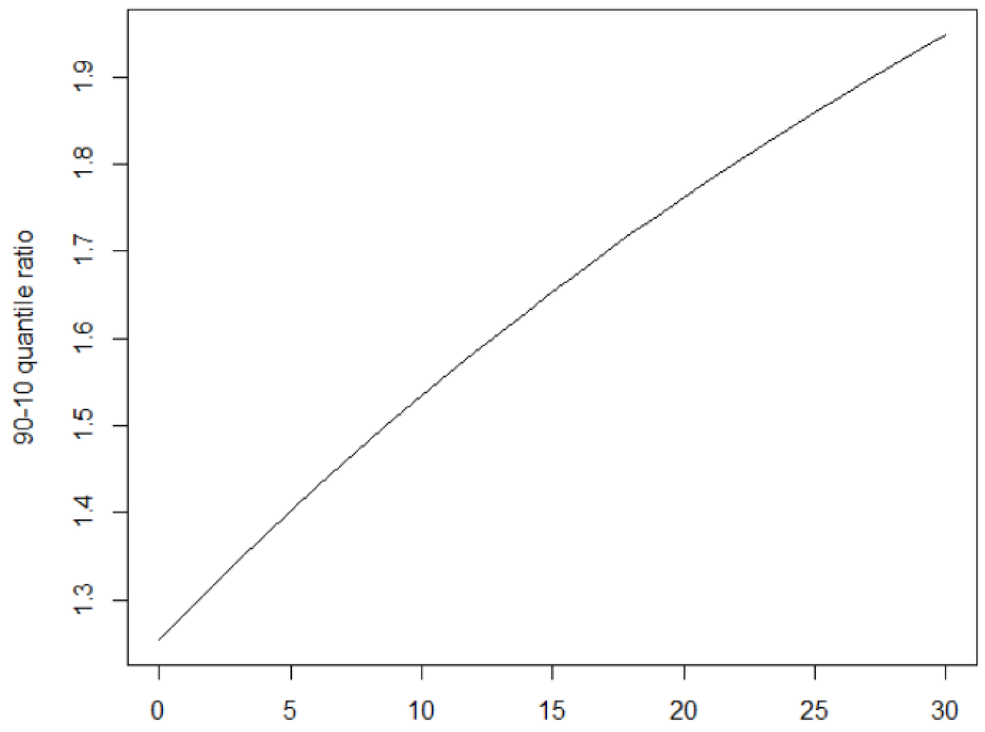
```
> u <- seq(.05,.95,by=.01)
> coefstd <- function(u) summary(
    rq(sl~yd,data=salary,tau=u))$
    coefficients[,2]
> coefest <- function(u) summary(
    rq(sl~yd,data=salary,tau=u))$
    coefficients[,1]
> CS <- Vectorize(coefstd)(u)
> CE <- Vectorize(coefest)(u)
> CEinf <- CE-2*CS
> CEsup <- CE+2*CS
> plot(u,CE[2,],ylim=c(-500,2000)
    ,col="red")
> polygon(c(u,rev(u)),c(CEinf
    [2,],rev(CEsup[2,])),col="
    yellow",border=NA)
```



Quantile Regression : Empirical Analysis

Consider the evolution of the 90% − 10% quantile ratio,

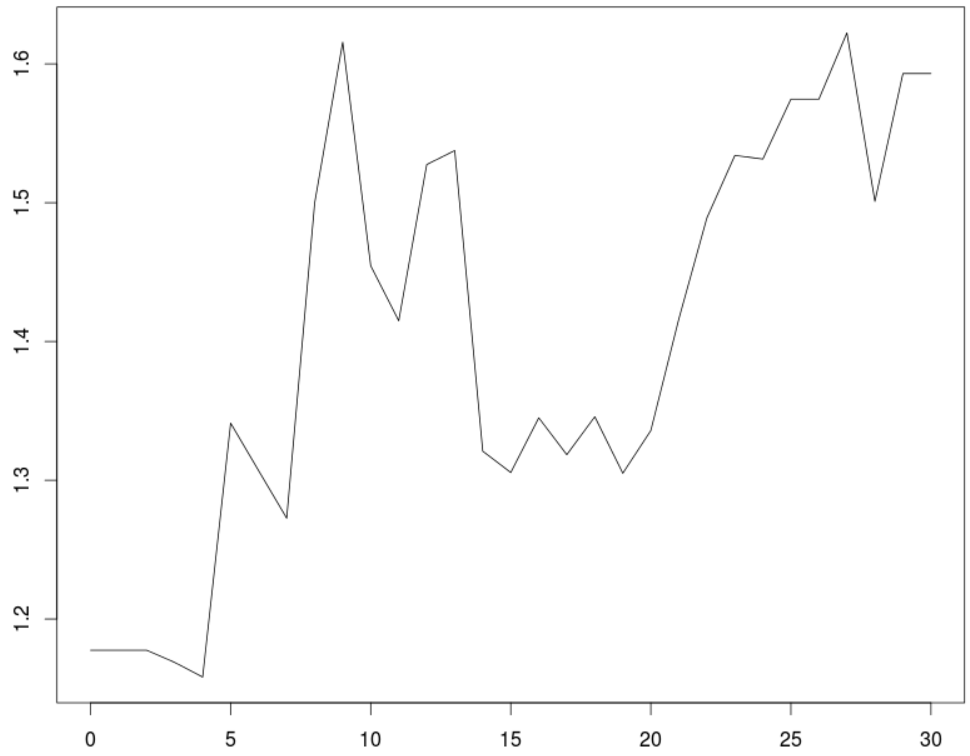
```
> ratio9010 = function(age){
+     predict(Q90,newdata=data.
      frame(yd=age))/
+     predict(Q10,newdata=data.
      frame(yd=age))
+ }
> ratio9010(5)
1.401749
> A=0:30
> plot(A,Vectorize(ratio9010)(A),
      type="l",ylab="90-10 quantile
       ratio")
```



Local Regression : Empirical Analysis

which is smoother than the local estimator

```
> ratio9010_k = function(age,k
    =10){
+     idx=which(rank(abs(salary$yd-
    age))<=k)
+     quantile(salary$sl[idx],.9)/
    quantile(salary$sl[idx],.1)}
> A=0:30
> plot(A,Vectorize(ratio9010_k)(A
    ),type="l",ylab="90-10
    quantile ratio")
```

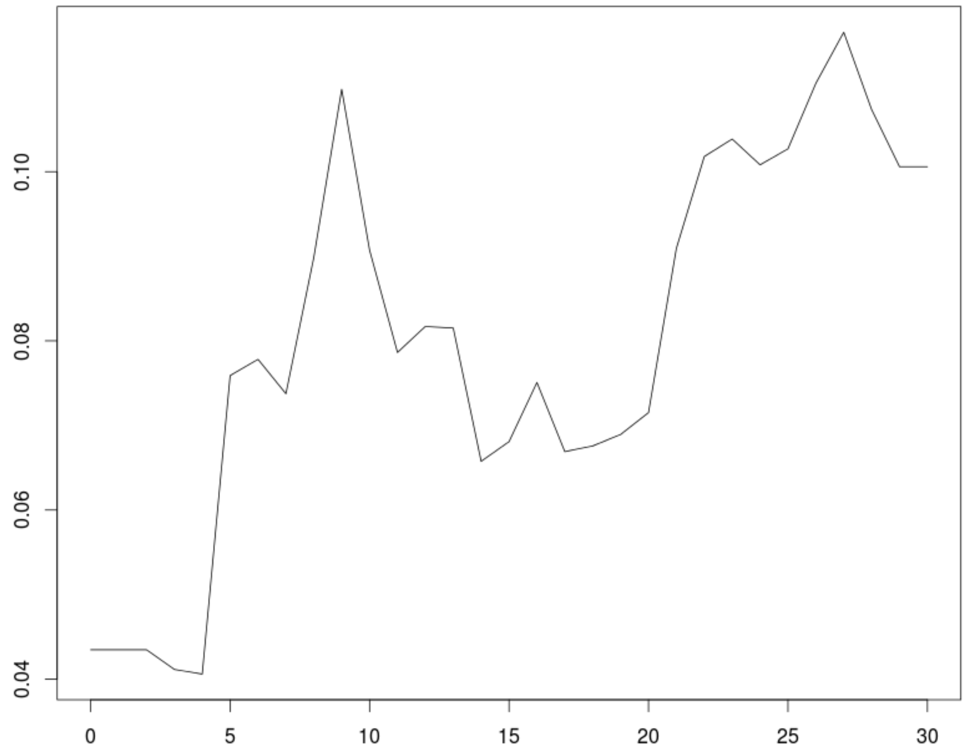


Local Regression : Empirical Analysis

```
> Gini(salary$sl)
[1] 0.1391865
```

We can also consider some local Gini index

```
> Gini_k = function(age,k=10){
+     idx=which(rank(abs(salary$yd-
      age))<=k)
+     Gini(salary$sl[idx])}
> A=0:30
> plot(A,Vectorize(Gini_k)(A),
      type="l",ylab="Local Gini
      index")
```



Datasets for Empirical Analysis

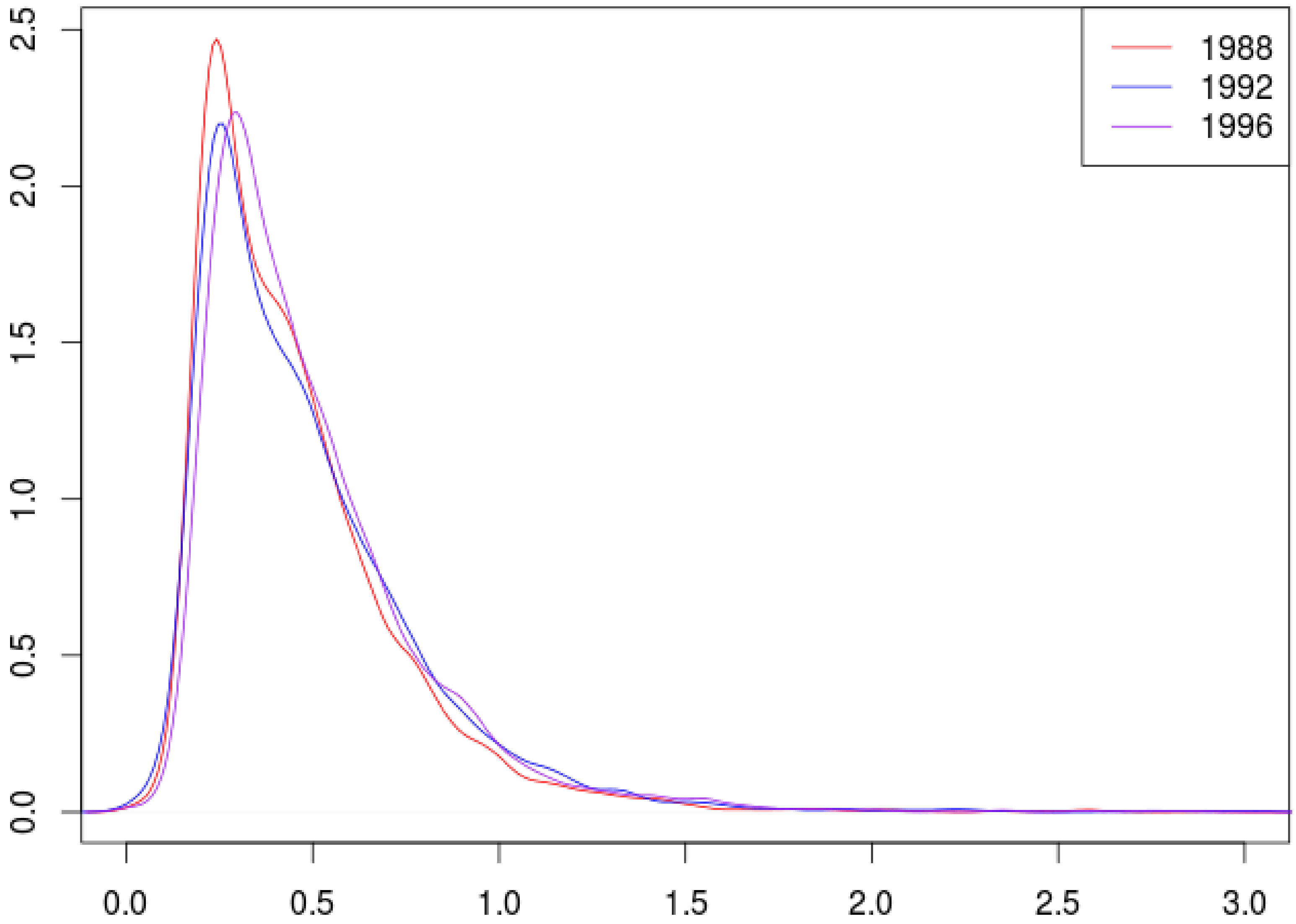
Income the U.K., in 1988, 1992 and 1996,

```
> uk88 <- read.csv("http://www.vcharite.univ-mrs.fr/pp/lubrano/cours/fes88.csv",sep=";",header=FALSE)$V1
> uk92 <- read.csv("http://www.vcharite.univ-mrs.fr/pp/lubrano/cours/fes92.csv",sep=";",header=FALSE)$V1
> uk96 <- read.csv("http://www.vcharite.univ-mrs.fr/pp/lubrano/cours/fes96.csv",sep=";",header=FALSE)$V1
> cpi <- c(421.7, 546.4, 602.4)

> y88 <- uk88/cpi[1]
> y92 <- uk92/cpi[2]
> y96 <- uk96/cpi[3]

> plot(density(y88),type="l",col="red")
> lines(density(y92),type="l",col="blue")
> lines(density(y96),type="l",col="purple")
```

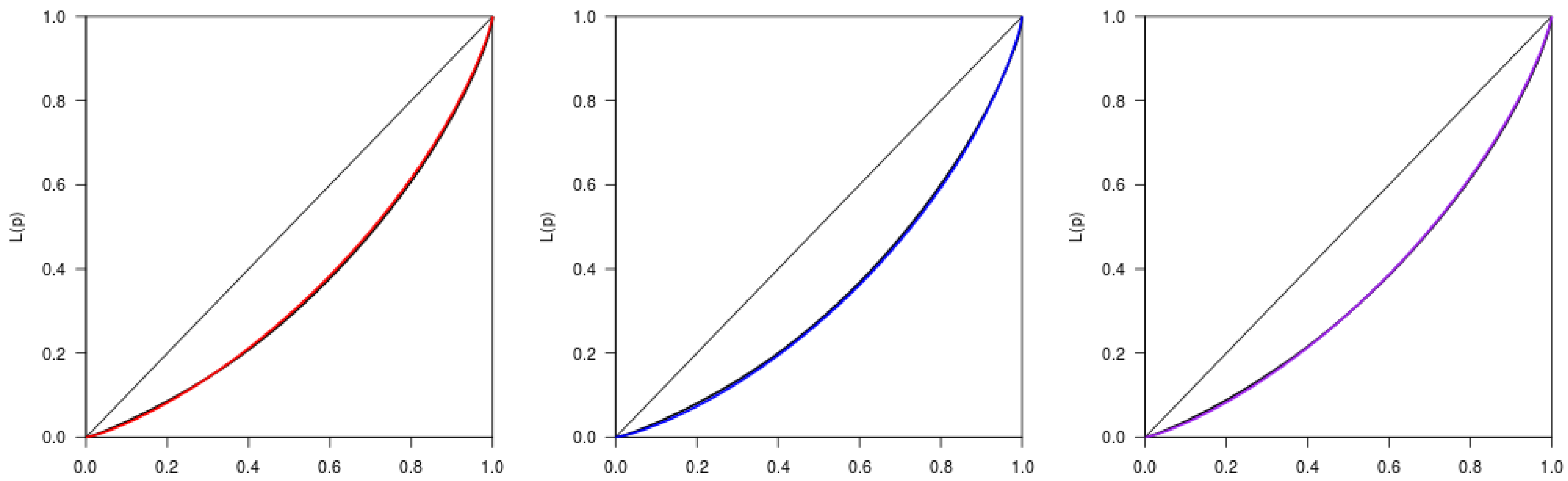
Datasets for Empirical Analysis



Inequalities : Empirical Analysis

We can visualize empirical Lorenz curves, and theoretical version (lognormal)

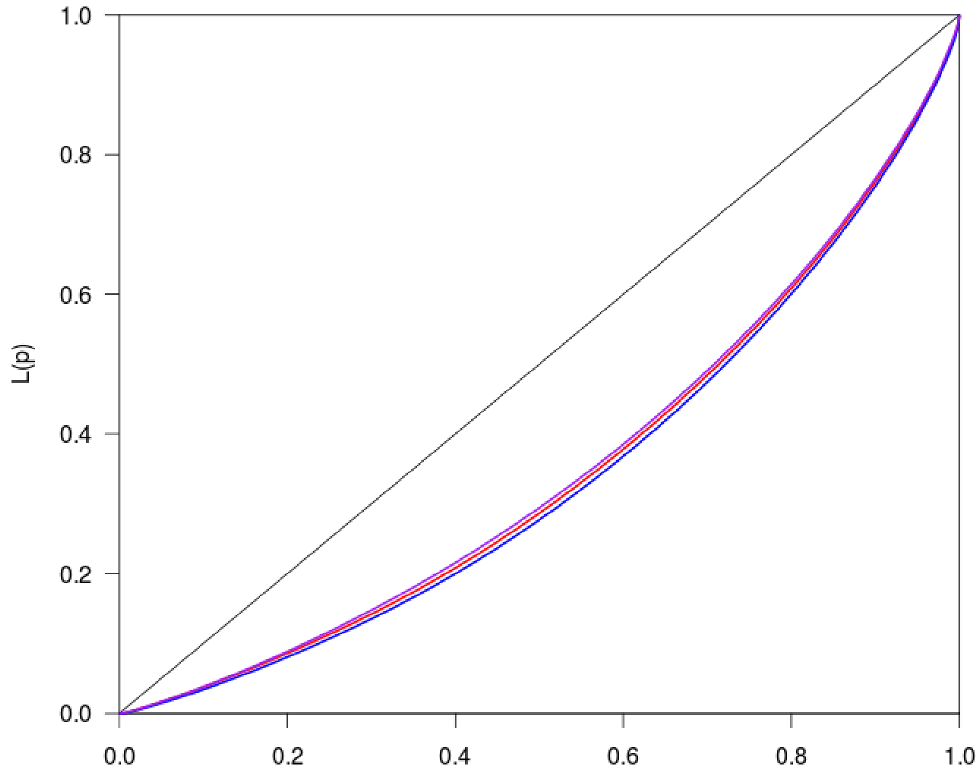
```
> plot(Lc(y88)); s=sd(log(y88)); lines(Lc.lognorm, parameter=s)
> plot(Lc(y92)); s=sd(log(y92)); lines(Lc.lognorm, parameter=s)
> plot(Lc(y96)); s=sd(log(y96)); lines(Lc.lognorm, parameter=s)
```



Inequalities : Empirical Analysis

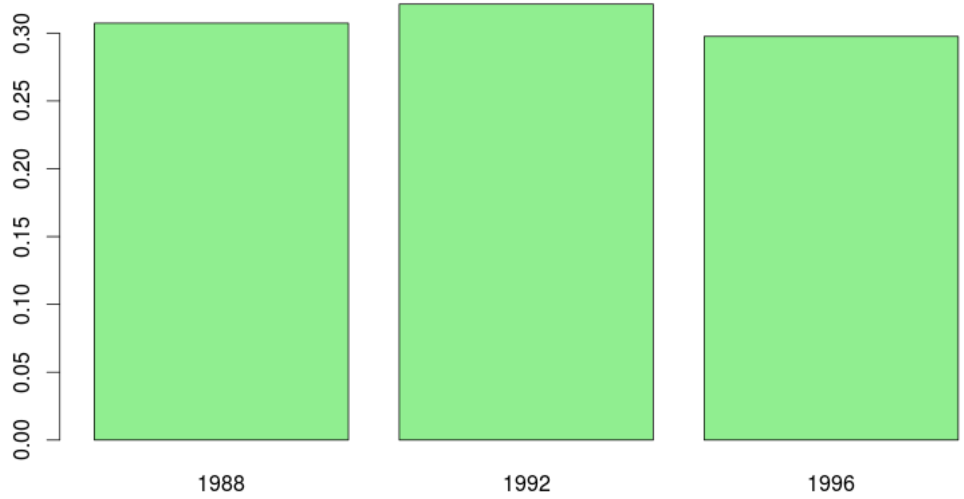
If we plot the three curves on the same graph,

```
> plot(Lc(y88),col="red")
> lines(Lc(y92),col="blue")
> lines(Lc(y96),col="purple")
```



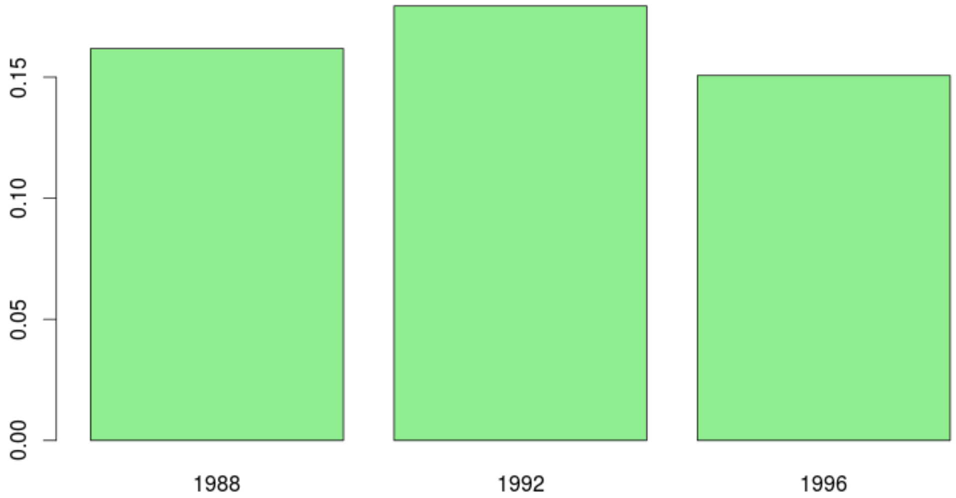
Inequalities : Empirical Analysis

```
> inequalities=function(f_ineq){
+   z88 =f_ineq(y88); z92 = f_ineq
    (y92); z96 = f_ineq(y96)
+   I=cbind(z88,z92,z96)
+   names(I)=c("1988","1992","1996
    ")
+   cat(" 1988...",z88,"\n 1992...
    ",z92,"\n 1996...",z96,"\n")
+   barplot(I,col="light green",
    names.arg=c("1988","1992","
    1996"))
+   return(I)}
> I<-inequalities(Gini)
 1988... 0.3073511
 1992... 0.3214023
 1996... 0.2975789
```



Inequalities : Empirical Analysis

```
> I<-inequalities(Theil)
  1988...  0.1618547
  1992...  0.1794008
  1996...  0.1506973
```



Welfare Functions

A welfare function as a function with n arguments $W(\boldsymbol{x}) = W(x_1, \cdots, x_n)$. Assume that W is normalize, so that $W(\mathbf{1}) = 1$.

It represents social preferences over the income distribution, and it should satisfy some axioms,

Pareto axiom : The welfare function is increasing for all its inputs

$$W(\boldsymbol{x} + \boldsymbol{\epsilon}) \geq W(\boldsymbol{x}) \text{ for all} \boldsymbol{\epsilon} \geq 0.$$

Symmetry axiom or anonymity : We can permute the individuals without changing the value of the function

$$W(x_1, x_2, \cdots, x_n) = W(x_2, x_1, \cdots, x_n)$$

Welfare Functions

Principle of transfers : the quasi concavity of the welfare function implies that if we operate a monetary transfer from a rich to a poor, welfare is increased, provided that the transfer does not modify the ordering of individuals (Pigou-Dalton principle)

$$W(x_1,\cdots,x_i,\cdots,x_j,\cdots,x_n) \geq W(x_1,\cdots,x_i+\delta,\cdots,x_j-\delta,\cdots,x_n)$$

Other axioms can be added, e.g. homogeneous of order 1,

$$W(\lambda\boldsymbol{x}) = \lambda W(\boldsymbol{x}) \text{ for all } \lambda \geq 0.$$

Thus (all homogeneous function of order 1 can be defined on the simplex)

$$W(\boldsymbol{x}) = \overline{x} \cdot W\left(\frac{\boldsymbol{x}}{\overline{x}}\right) \text{ for all } \lambda \geq 0.$$

Welfare Functions

Observe that $W(\overline{x}\mathbf{1}) = \overline{x}$. And because of the aversion for inequality, $W(\boldsymbol{x}) \leq \overline{x}$. One can denote

$$W(\boldsymbol{x}) = \overline{x} \cdot [1 - I(\boldsymbol{x})]$$

for some function $I(\cdot)$, which takes values in $[0, 1]$.

$I(\cdot)$ is then interpreted as an inequality measure and $\overline{x} \cdot I(\boldsymbol{x})$ represents the (social) cost of inequality.

See fao.org.

Welfare Functions

E.g. utilitarian (or Benthamian) function

$$W(\boldsymbol{x}) = \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

"*dollar is a dollar*" approach : no inequality aversion.

E.g. Rawlsian welfare function

$$W(\boldsymbol{x}) = \min\{y_1, \cdots, y_n\}.$$

Social welfare cannot increase unless the income of the poorest individual is increased : infinite inequality aversion.

From Welfare Functions to Inequality Indices

Consider the standard welfare function,

$$W(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^{1-\epsilon}}{1-\epsilon}$$

with the limiting case (where $\epsilon \to 1$)

$$W(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n}\log(x_i)$$

When $\epsilon \to 1$ we have the Benthamian function, and when $\epsilon \to \infty$, we have the Rawlsian function. Thus, ε can be interpreted as an inequality aversion parameter.

The ratio of marginal social utilities of two individuals i and j has a simple expression

$$\frac{\partial W/\partial x_i}{\partial W/\partial x_j} = \left(\frac{x_i}{x_j}\right)^{-\epsilon}$$

When ϵ increases, the marginal utility of the poorest dominates, see Rawls (1971) wikipedia.org, the objective of the society is to maximise the situation of the poorest.

From that welfare function, define the implied inequality index,

$$I = 1 - \left(\frac{1}{n} \sum_{i=1}^{n} \left(\frac{x_i}{\overline{x}} \right)^{1-\epsilon} \right)^{\frac{1}{1-\varepsilon}}$$

which is Atkinson index.

Equally Distributed Equivalent

Given $\boldsymbol{x}$ define ξ (or $\xi(\boldsymbol{x})$) as

$$W(\xi\mathbf{1}) = W(\boldsymbol{x})$$

From the principle of transfers, $\xi \leq \overline{x}$. Then one can define

$$I(\boldsymbol{x}) = 1 - \frac{\xi(\boldsymbol{x})}{\overline{x}}.$$

If $I(\cdot)$ satisfies the scale independence axiom, $I(\boldsymbol{x}) = I(\lambda\boldsymbol{x})$, then

$$\xi(\boldsymbol{x}) = \left(\frac{1}{n}\sum_{i=1}^{n}(x_i)^{1-\epsilon}\right)^{\frac{1}{1-\varepsilon}}$$

This index has a simple interpretation : if $I = 0.370\%$ of the total income is necessary to reach the same value of welfare, provided that income is equally distributed.

Kolm (1976) suggested that the welfare function should not change if the same positive amount is given to everybody, i.e.

$$W(\boldsymbol{x}) = W(\boldsymbol{x} + h\mathbf{1})$$

This leads to

$$I(\boldsymbol{x}) = \frac{1}{\alpha} \log\left(\frac{1}{n} \sum_{i=1}^{n} \exp[\alpha(x_i - \overline{x})] \right)$$

From Inequality Indices to Welfare Functions

Consider e.g. Gini index

$$G(\boldsymbol{x}) = \frac{2}{n(n-1)\overline{x}} \sum_{i=1}^{n} i \cdot x_{i:n} - \frac{n+1}{n-1}$$

$$G(\boldsymbol{x}) = \frac{1}{2n^2\overline{x}} \sum_{i,j=1}^{n} |x_i - x_j|$$

then define

$$W(\boldsymbol{x}) = \overline{x} \cdot [1 - G(\boldsymbol{x})]$$

as suggested in Sen (1976, jstor.org)

More generally, consider

$$W(\boldsymbol{x}) = \overline{x} \cdot [1 - G(\boldsymbol{x})]^{\sigma}$$

with $\sigma \in [0, 1]$.

From Inequality to Poverty

an absolute line of poverty is defined with respect to a minimum level of subsistence

In developed countries and more precisely within the EU, o ne prefer to define a relative poverty line, defined with respect to a fraction of the mean or the median of the income distribution.

The headcount ratio evaluates the number of poor (below a threshold z)

$$H(\boldsymbol{x},z)=\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(x_i\leq z)=\widehat{F}(z)=\frac{q}{n}$$

where q is the number of poors.

The income gap ratio $I(x,z)$ measures in percentage the gap between the poverty line z and the mean income among the poor

$$I(\boldsymbol{x},z)=\frac{1}{z}\left(z-\frac{1}{q}\sum_{i=1}^{n}x_i\mathbf{1}(x_i\leq z)\right)=\frac{1}{z}\left(z-\frac{1}{q}\sum_{i=1}^{q}x_{i:n}\right)=1-\frac{\mu_p}{z}$$

where μ_p is the average income of the poor.

The poverty gap ratio is defined as

$$HI(\boldsymbol{x},z)=\frac{q}{n}\left(1-\frac{1}{qz}\sum_{i=1}^{q}x_{i:n}\right)$$

Watts (1968) suggested also

$$W(\boldsymbol{x},z)=\frac{1}{q}\sum_{i=1}^{q}[\log z-\log x_{i:n}]$$

which can be writen

$$W=H\cdot(T-\log(1-I))$$

where T is Theil index (Generalize Entropy, with index 1)

$$T=\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{\overline{x}}\log\left(\frac{x_i}{\overline{x}}\right).$$

```
> Watts(x, z, na.rm = TRUE)
```

Sen Poverty Indices

$$S(\boldsymbol{x},z)=H(\boldsymbol{x},z)\cdot[I(\boldsymbol{x},z)+[1-(\boldsymbol{x},z)]G_p]$$

where G_p is Gini index of the poors.

— if $G_p=0$ then $S=HI$

— if $G_p=1$ then $S=H$

```
> Sen(x, z, na.rm = TRUE)
```

On can write

$$S=\frac{2}{(q+1)nz}\sum_{i=1}^{q}[z-x_{i:n}][q+1-i]$$

Thon (1979) suggested a similar expression, but with (slightly) different weights

$$\text{Thon}=\frac{2}{n(n+1)z}\sum_{i=1}^{q}[z-x_{i:n}][n+1-i]$$

But it suffers some drawbacks : it violates the principle of transfers and is not continuous in $\boldsymbol{x}$. Shorrocks (1995, jstor.org) suggested

$$SST(\boldsymbol{x},z) = [2 - H(\boldsymbol{x},z)] \cdot H(\boldsymbol{x},z) \cdot I(\boldsymbol{x},z) + H(\boldsymbol{x},z)^2[1 - I(\boldsymbol{x},z)] \cdot G_P$$

Observe that Sen index is defined as

$$S = \frac{2}{(q+1)n} \sum_{i=1}^{q} \underbrace{\frac{z - x_{i:n}}{z}}_{\tilde{x}_i} [q+1-i]$$

while

$$SST = \frac{1}{n^2} \sum_{i=1}^{q} \underbrace{\frac{z - x_{i:n}}{z}}_{\tilde{x}_i} [2n - 2i + 1]$$

This index is symmetric, monotonic, homogeneous of order 0 and takes values in $[0,1]$. Further it is continuous and consistent with the transfert axiom.

On can write

$$SST = \overline{\tilde{x}} \cdot [1 - G(\tilde{\boldsymbol{x}})].$$

```
> SST(x, z, na.rm = TRUE)
```

```
> poverty=function(f_pov,z_fun=function(x) mean(x)/2,...){
+   z88 =z_fun(y88); z92 = z_fun(y92); z96 = z_fun(y96)
+   p88=f_pov(y88,z88); p92=f_pov(y92,z92); p96=f_pov(y96,z96)
+   P=cbind(p88,p92,p96)
+   names(P)=c("1988","1992","1996")
+   cat(" 1988...",p88,"\n 1992...",p92,"\n 1996...",p96,"\n")
+   barplot(P,col="light green",names.arg=c("1988","1992","1996"))
+   return(P)}
```

FGT Poverty Indices

Foster, Greer & Thorbecke (1984, darp.lse.ac.uk) suggested a class of poverty indices that were decomposable,

$$P_\alpha(\boldsymbol{x}, z) = \frac{1}{n}\sum_{i=1}^{q}\left(1 - \frac{x_i}{z}\right)^\alpha$$

where $\alpha \in \{0, 1, 2, \cdots\}$.

When $\alpha = 0$ we get the headcount measure,

$$P_0 = \frac{1}{n}\sum_{i=1}^{q}\mathbf{1}(x_i \leq z) = \frac{q}{n}$$

When $\alpha = 1$ we get an average of poverty gap $z - x_i$

$$P_1 = \frac{1}{n}\sum_{i=1}^{q}\left(1 - \frac{x_i}{z}\right)\mathbf{1}(x_i \leq z)$$

(see HI).

In R, the parameter is $1+\alpha$

```
> Foster(x, k, parameter = 1, na.rm = TRUE)
```

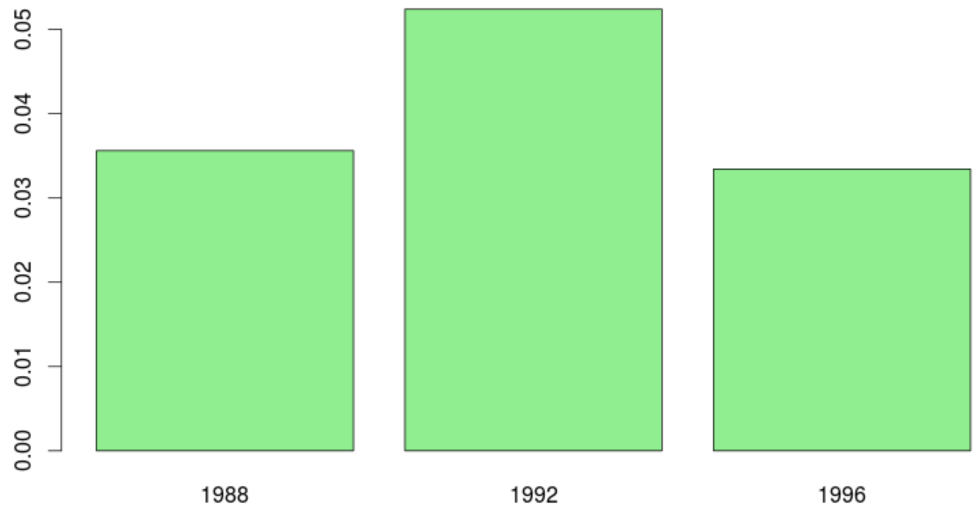
i.e. it gives for parameter 1 the headcount ratio and for parameter 2 the poverty gap ratio.

When $\alpha = 2$

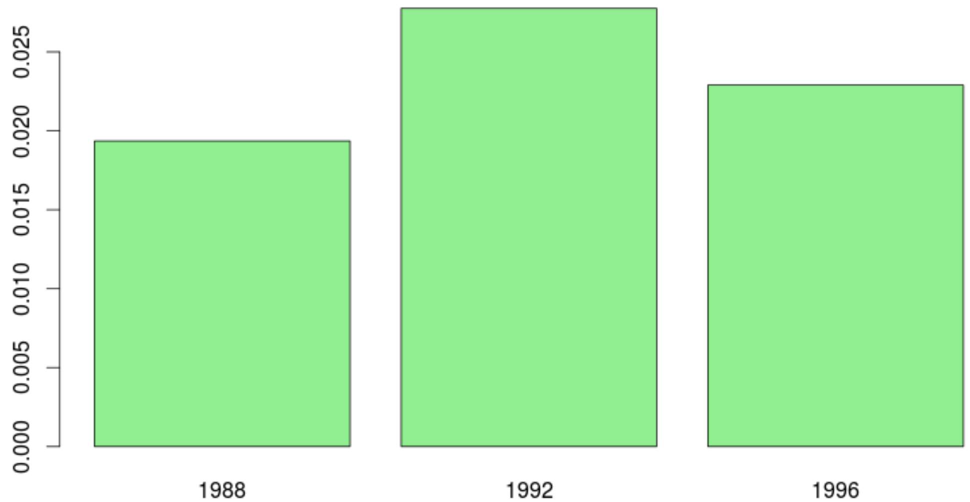
$$P_2 = \frac{1}{n}\sum_{i=1}^{q}\left(1-\frac{x_i}{z}\right)^2 \mathbf{1}(x_i \leq z)$$

Inequalities : Empirical Analysis

```
> P<-poverty(Watts,function(x)
      mean(x)/2)
 1988...  0.03561864
 1992...  0.05240638
 1996...  0.03342492
```

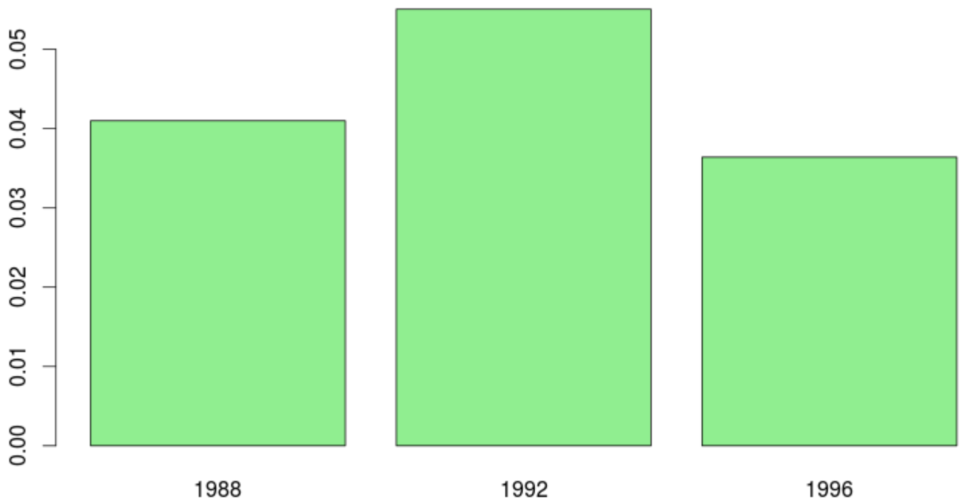



```
> P<-poverty(Watts,function(x)
      quantile(x,.1))
 1988...  0.01935494
 1992...  0.0277594
 1996...  0.02289631
```

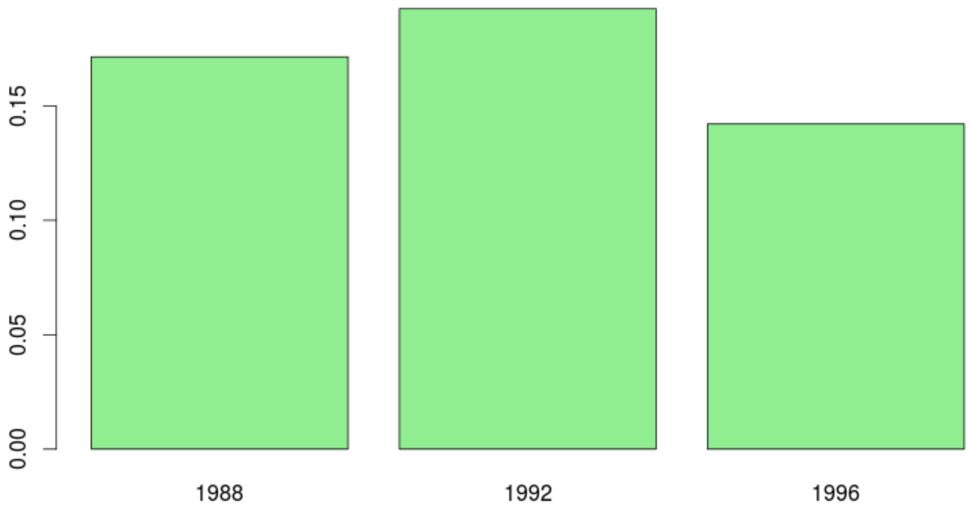


Inequalities : Empirical Analysis

```
> P<-poverty(Sen,function(x) mean
    (x)/2)
  1988...  0.04100178
  1992...  0.05507059
  1996...  0.03640762
```




```
> P<-poverty(Foster,function(x)
    mean(x)/2,param=0)
  1988...  0.1714684
  1992...  0.1925117
  1996...  0.1421479
```



Group Decomposabilty

Assume that x is either x_1 with probability p (e.g. *urban*) or x_2 with probability $1-p$ (e.g. *rural*). The (total) FGT index can be writen

$$P_\alpha = p \cdot \frac{1}{n} \sum_{i,1} \left(1 - \frac{x_i}{z}\right)^\alpha + [1-p] \cdot \frac{1}{n} \sum_{i,2} \left(1 - \frac{x_i}{z}\right)^\alpha = pP_\alpha^{(1)} + [1-p]P_\alpha^{(2)}$$

Welfare, Poverty and Inequality

Atkinson (1987, darp.lse.ac.uk) suggested several options,

- neglect poverty, $W(\boldsymbol{x}) = \overline{x} \cdot [1 - I(\boldsymbol{x})]$,
- neglect inequality, $W(\boldsymbol{x}) = \overline{x} \cdot [1 - P(\boldsymbol{x})]$,
- tradeoff inequality - poverty, $W(\boldsymbol{x}) = \overline{x} \cdot [1 - I(\boldsymbol{x}) - P(\boldsymbol{x})]$,