# Actuariat de l'Assurance Non-Vie # 10

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credit: Arnold Odermatt

## Généralités sur les Provisions pour Sinistres à Payer

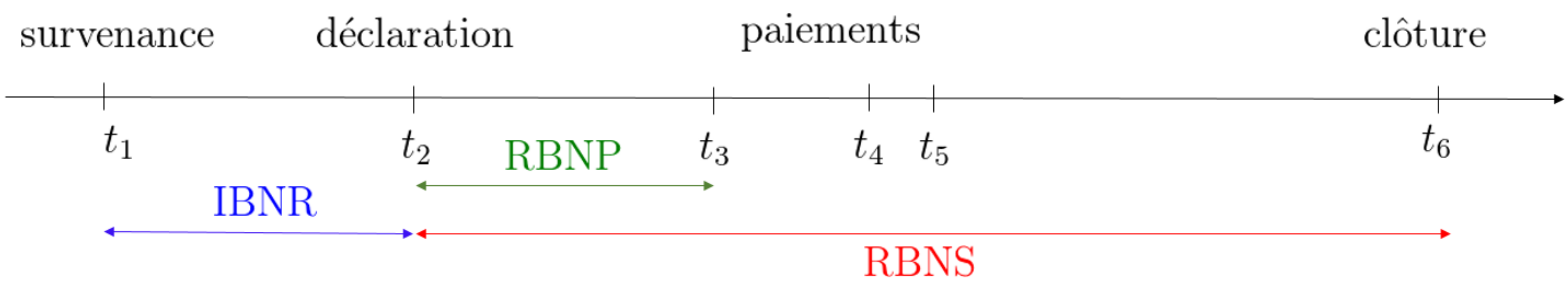
**Références**: de Jong & Heller (2008), section 1.5 et 8.1, and Wüthrich & Merz (2015), chapitres 1 à 3, et Pigeon (2015).

“ *Les provisions techniques sont les provisions destinées à permettre le réglement intégral des engagements pris envers les assurés et bénéficaires de contrats. Elles sont liées à la technique même de l'assurance, et imposées par la réglementation.*”

“*It is hoped that more casualty actuaries will involve themselves in this important area. IBNR reserves deserve more than just a clerical or cursory treatment and we believe, as did Mr. Tarbell Chat 'the problem of incurred but not reported claim reserves is essentially actuarial or statistical'. Perhaps in today's environment the quotation would be even more relevant if it stated that the problem '...is more actuarial than statistical'.*” Bornhuetter & Ferguson (1972)

## La vie des sinistres

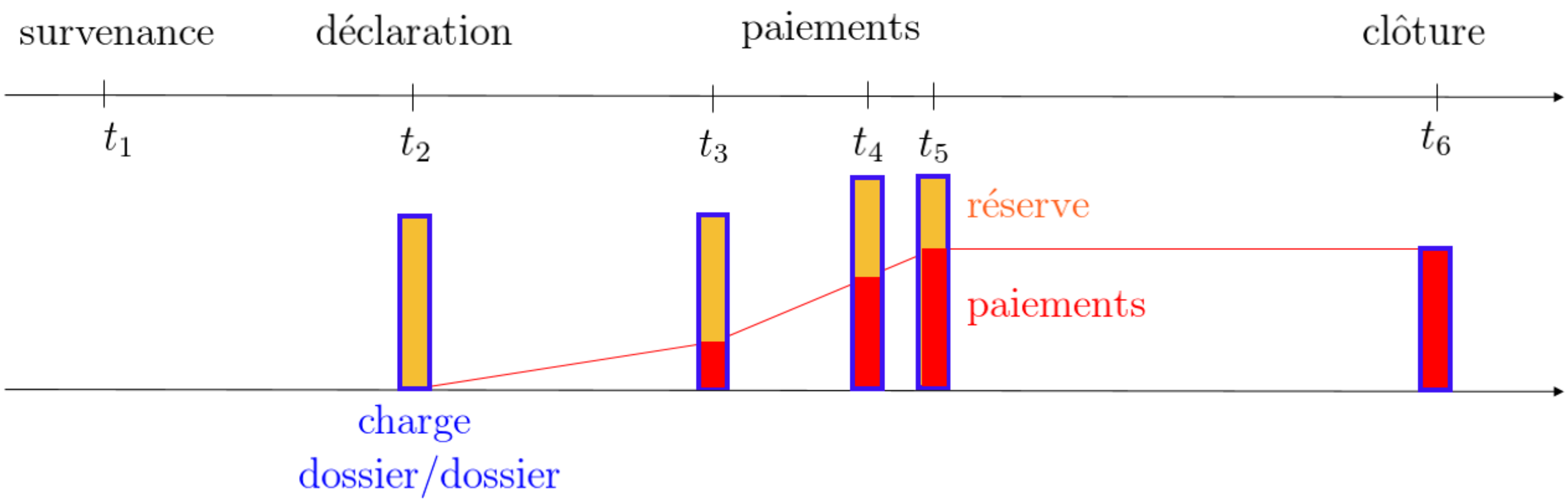
- date $t_1$ : survenance du sinistre
- période $[t_1, t_2]$ : IBNR Incureed But Not Reported
- date $t_2$ : déclaration du sinistre à l'assureur
- période $[t_2, t_3]$ : IBNP Incureed But Not Paid
- période $[t_2, t_6]$ : IBNS Incureed But Not Settled
- dates $t_3, t_4, t_5$ : paiements
- date $t_6$ : clôture du sinistre



## La vie des sinistres

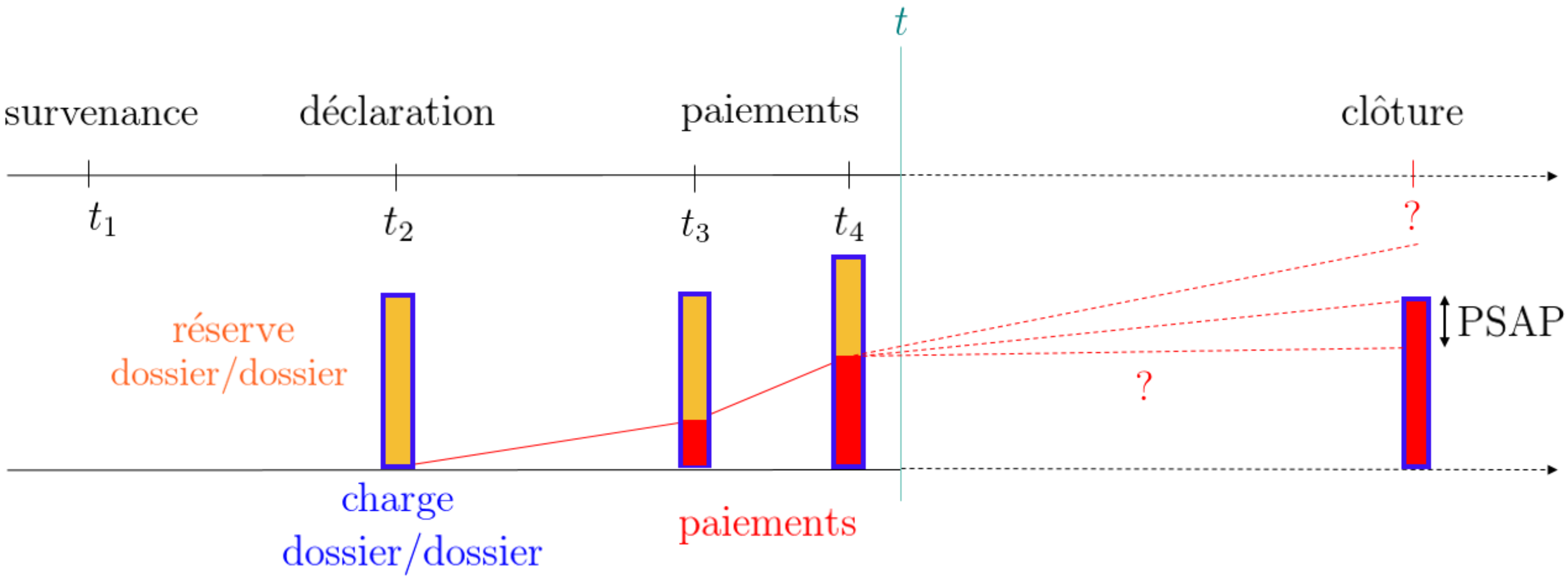
À la survenance, un montant estimé est indiqué par les gestionnaires de sinistres (charge dossier/dossier). Ensuite deux opérations sont possibles :

- effectuer un paiement
- réviser le montant du sinistre

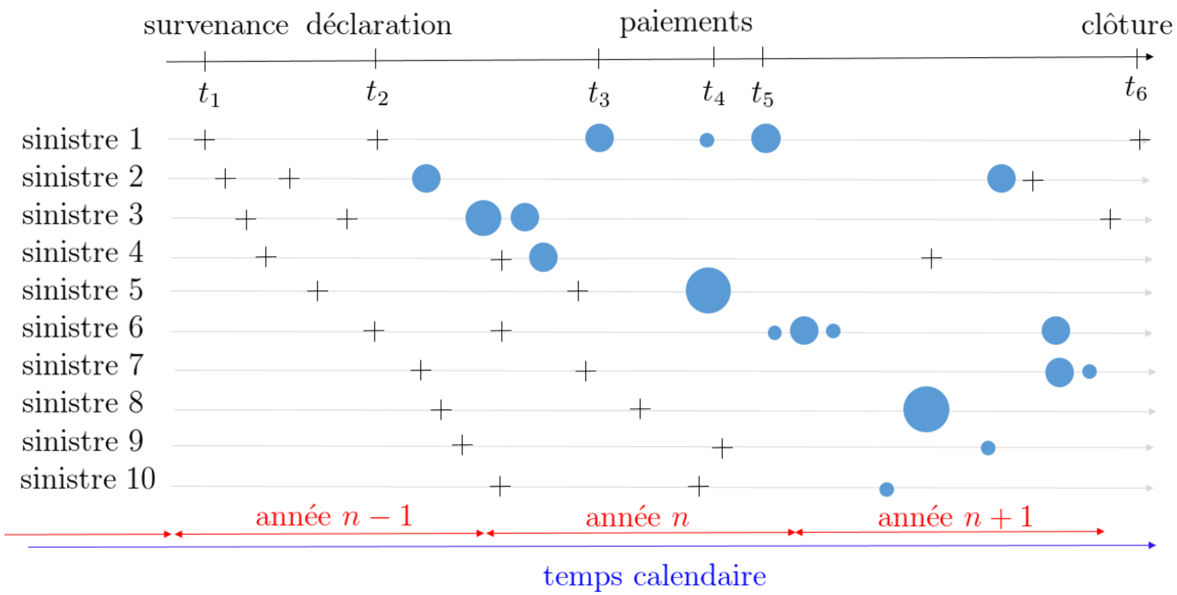


## La vie des sinistres: Problématique des PSAP

La Provision pour Sinistres à Payer est la différence entre le montant du sinistre et le paiement déjà effectué.



## Triangles de Paiements : du micro au macro




Analyse micro

Sinistres $i = 1, \cdots, n$

survenances, date $T_{i,0}$

déclaration, date $T_{i,0} + Q_i$

paiements, dates

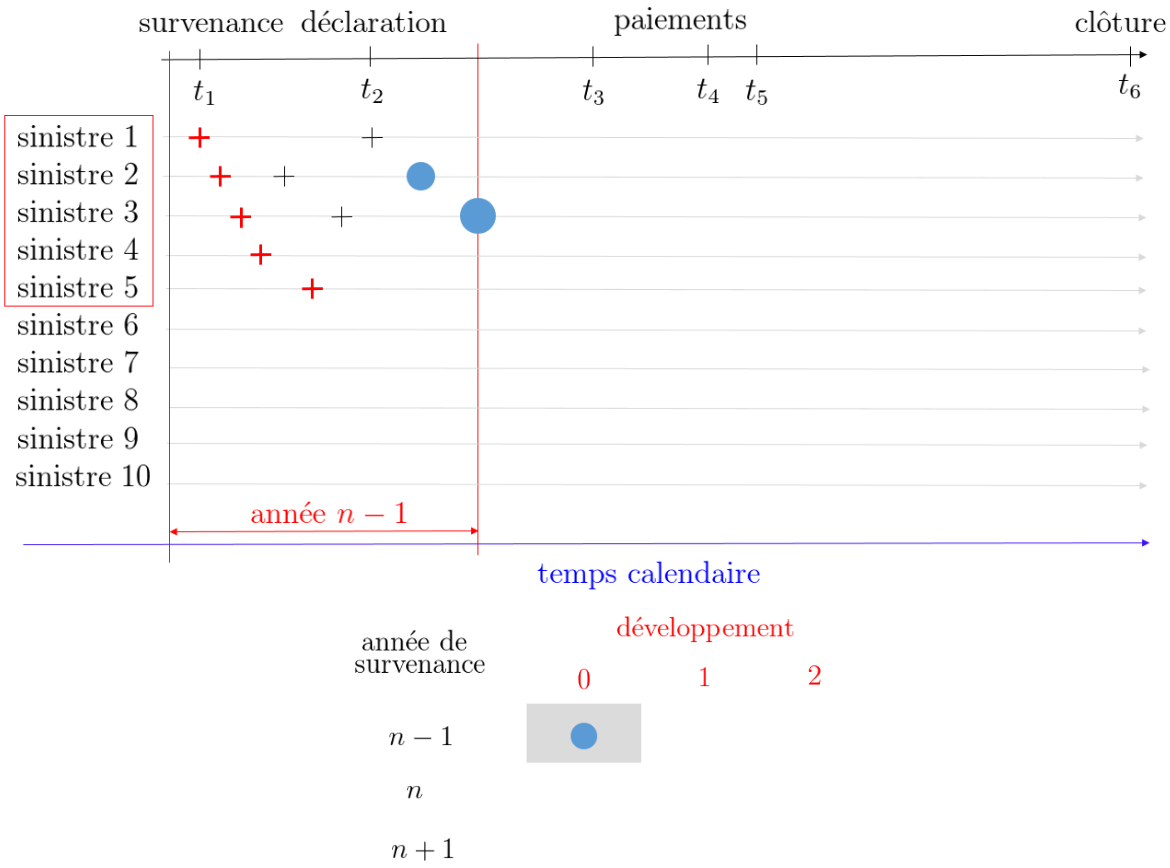
$$T_{i,j} = T_{i,0} + Q_i + \sum_{k=1}^{j} Z_{i,k}$$

et montants $(T_{i,j}, Y_{i,j})$

Montant de $i$ à la date $t$, $C_i(t) = \displaystyle\sum_{j:T_{i,j} \leq t} Y_{i,j}$

Provision (idéale) à la date $t$, $R_i(t) = C_i(\infty) - C_i(t)$

## Triangles de Paiements : du micro au macro




Analyse macro

On agrége les sinistres par année de survenance :

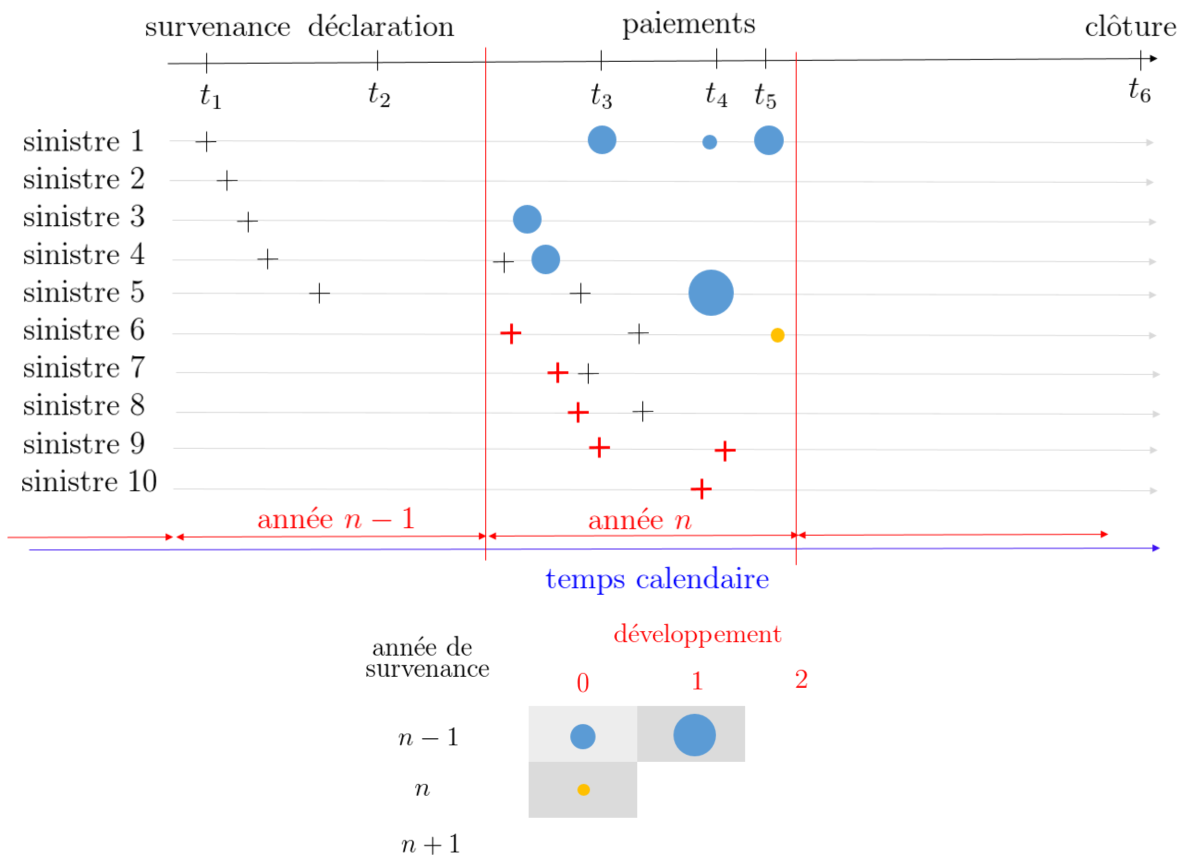
$$\mathcal{S}_i = \{k : T_{k,0} \in [i, i+1)\}$$

On regarde les paiements effectués après $j$ années

$$Y_{i,j} = \sum_{k \in \mathcal{S}_i} \sum_{T_{k,\ell} \in [j,j+1)} Z_{k,\ell}$$

Ici $Y_{n-1,0}$

## Triangles de Paiements : du micro au macro




Analyse macro

On agrége les sinistres par année de survenance :

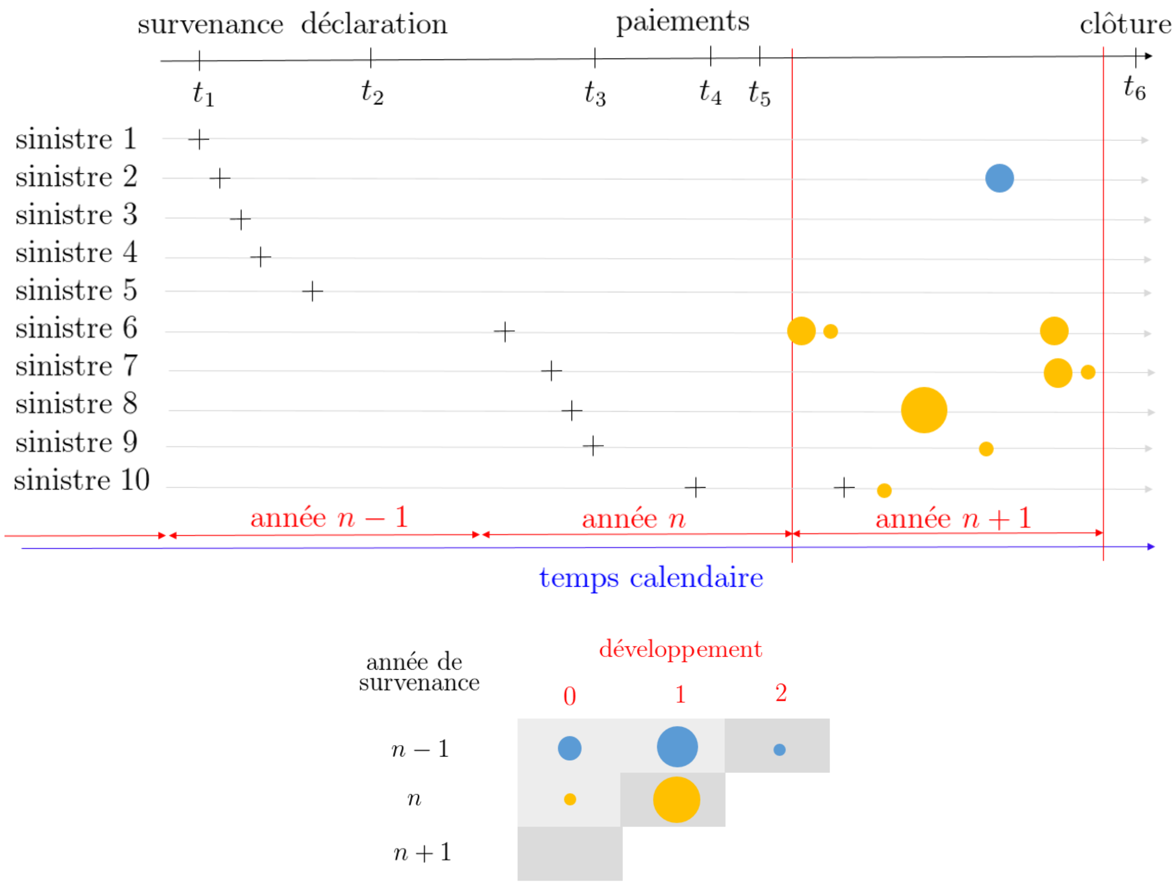
$$\mathcal{S}_i = \{k : T_{k,0} \in [i, i+1)\}$$

On regarde les paiements effectués après $j$ années

$$Y_{i,j} = \sum_{k \in \mathcal{S}_i} \sum_{T_{k,\ell} \in [j,j+1)} Z_{k,\ell}$$

Ici $Y_{n-1,1}$ et $Y_{n,0}$

## Triangles de Paiements : du micro au macro




Analyse macro

On agrége les sinistres par année de survenance :

$$\mathcal{S}_i = \{k : T_{k,0} \in [i, i+1)\}$$

On regarde les paiements effectués après $j$ années

$$Y_{i,j} = \sum_{k\in\mathcal{S}_i} \sum_{T_{k,\ell}\in[j,j+1)} Z_{k,\ell}$$

Ici $Y_{n-1,2}$, $Y_{n,1}$ et $Y_{n+1,0}$

## Triangles de Paiements : du micro au macro

- les provisions techniques peuvent représenter 75% du bilan,
- le ratio de couverture (provision / chiffre d'affaire) peut dépasser 2,
- certaines branches sont à développement long, en montant

| | $n$ | $n+1$ | $n+2$ | $n+3$ | $n+4$ |
|---|---|---|---|---|---|
| habitation | 55% | 90% | 94% | 95% | 96% |
| automobile | 55% | 80% | 85% | 88% | 90% |
| *dont corporels* | 15% | 40% | 50% | 65% | 70% |
| R.C. | 10% | 25% | 35% | 40% | 45% |

## Triangles de Paiements : complément sur les données micro

On note $\mathcal{F}_t^n$ l'information accessible à la date $t$,

$$\mathcal{F}_t^n = \sigma\left\{(T_{i,0}, T_{i,j}, X_{i,j}), i = 1, \cdots, n, T_{i,j} \leq t\right\}$$

et $C(\infty) = \sum_{i=1}^{n} C_i(\infty)$.

Notons que $M_t = \mathbb{E}\left[C(\infty)|\mathcal{F}_{\sqcup}\right]$ est une martingale, i.e.

$$\mathbb{E}[M_{t+h}|\mathcal{F}_t] = M_t$$

alors que $V_t = \text{Var}\left[C(\infty)|\mathcal{F}_{\sqcup}\right]$ est une sur-martingale, i.e.

$$\mathbb{E}[V_{t+h}|\mathcal{F}_t] \leq V_t.$$

# Triangles de Paiements : les méthodes usuelles

| | ODP/Bootstrap | Mack | Bayesian/BF Method | Judgement | Scenarios | Regression/Curve Fitting |
|---|---|---|---|---|---|---|
| **Description** | Most common bootstrap model. Potential to use different distribution for the residuals | Calculation of standard error with and without tail factors. | Uses ODP model with a series of prior ULR estimates defined by a distribution | Based on professional experience | Can include any variation such as changing development patterns or single events | Fits Craighead curve to each origin year to derive initial estimate of ULR, then smoothes across origin years using regression |
| **Data required** | Cumulative claims triangles (paid or incurred) | Cumulative claim triangles (paid or incurred) | Cumulative claim triangles (paid or incurred) | Any | Any | Premium and claim amounts triangles |
| **Is the method acceptable to the Profession?** | Yes | Yes | Yes | Yes | Yes | Depends on purpose |
| **Is the method easy to use and is it practical?** | Yes | Yes | No | Yes | Yes | Yes |
| **Can judgement or amendments be applied?** | Yes | Amendments needed where gaps in published method | Requires prior distribution of ultimate position of each origin year | Yes - essential | Yes via choice of scenarios and manual adjustments or tweaks | Yes, perhaps too easily |
| **Is the method easy to explain?** | Principles easy to explain | No | Very difficult | Yes | Yes | Yes |
| **When is method good? (Or not?)** | Good if little negative development and residuals are iid and run-off pattern is same for all years. | Good only if run-off pattern is same for all years | Good if little negative development and residuals are iid and run-off pattern is same for all years. | Good if actuary has additional knowledge; bad if not experienced | Not good if volatile datasets or inexperienced actuary | Good if run-off pattern varies across origin years. Not good if there is much negative development |

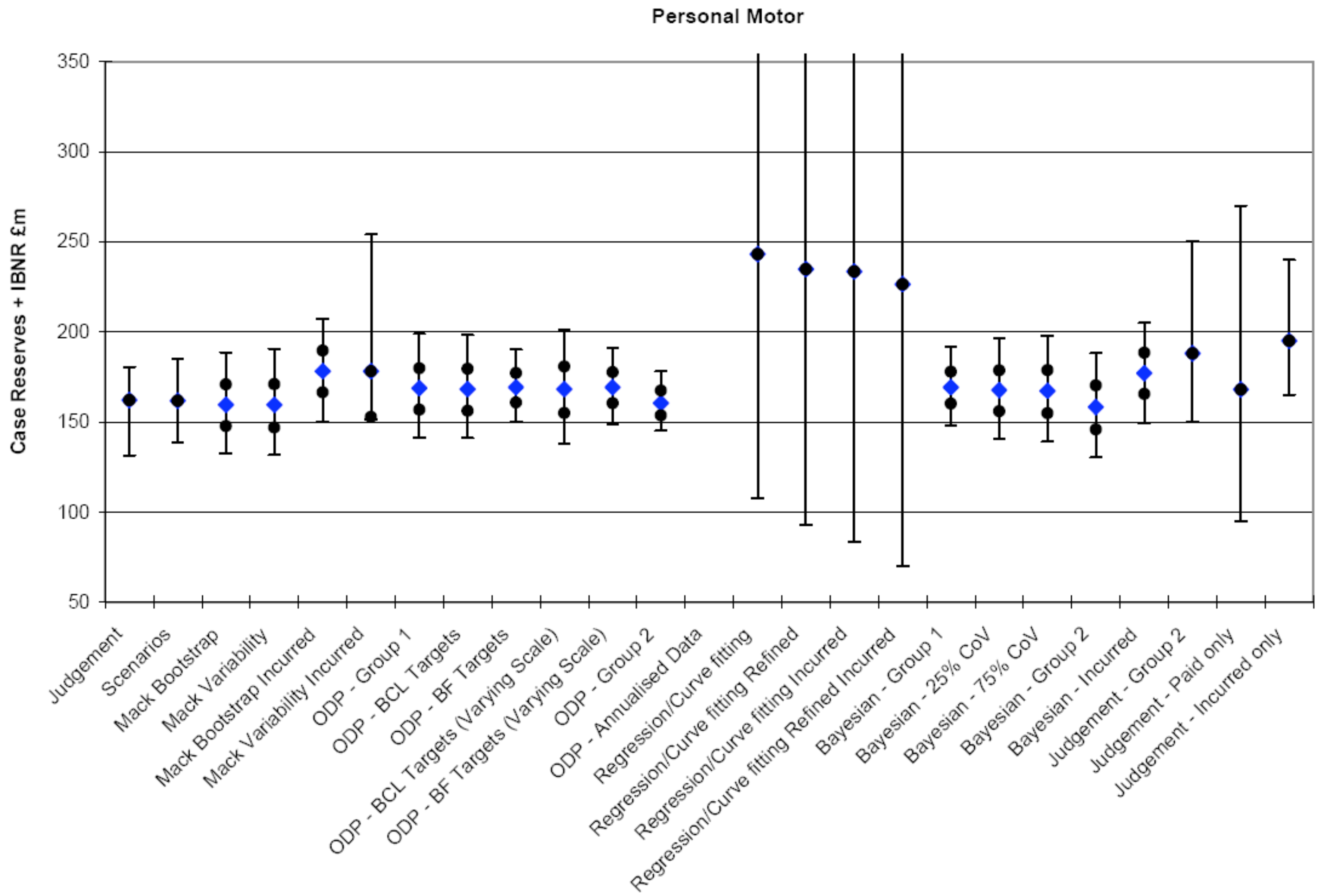
Source : http://www.actuaries.org.uk

# Triangles de Paiements : les méthodes usuelles

| | ODP/Bootstrap | Mack | Bayesian/BF Method | Judgement | Scenarios | Regression/Curve Fitting |
|---|---|---|---|---|---|---|
| **Are extreme events included?** | Only if in data | Only if in data | Yes, if in data and/or in prior distributions | Yes if desired | Yes if desired | Yes, if in data but can exclude if desired |
| **Produce complete distribution of outcomes?** | Yes if process error is simulated in addition to bootstrapping for parameter error | Produces mean and standard error only | Yes | Yes as any required percentile can be estimated using judgement | No – produces a few possible outcomes to which probabilities can be judgementally applied | No, just an approximate range |
| **Type of uncertainty measured** | Bootstrap method gives parameter uncertainty, process uncertainty can be simulated in addition | Process and parameter uncertainty | Process and parameter uncertainty | Potentially model error as well as parameter and process error | Usually just parameter uncertainty | Parameter uncertainty only (dependent variable in regression is expected ULR) |
| **Time to program and complete** | Easy to program in Excel though long time to run | Easy to program and quick to run | Specialist software required and very slow to run | | | Easy to do in Excel |
| **Comparison of class results to aggregated** | Automatic consistency between origin year and aggregate results | Automatic consistency between origin year and aggregate results | Automatic consistency between origin year and aggregate results | Should be consistent given enough care, but this not guaranteed | Does not produce separate assessment of aggregate uncertainty | Does not produce separate assessment of aggregate uncertainty |

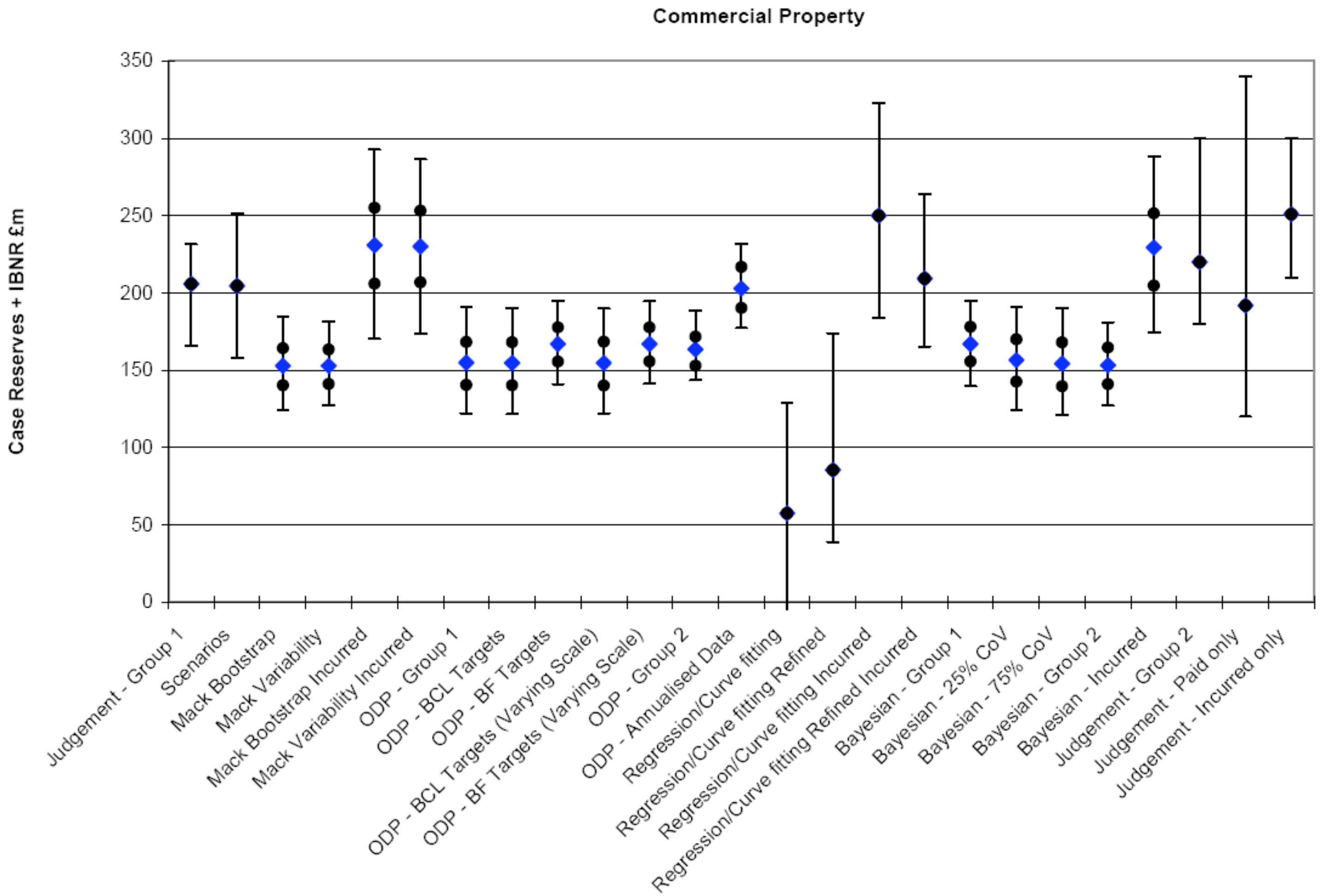
Source : http://www.actuaries.org.uk

# Triangles de Paiements : les méthodes usuelles



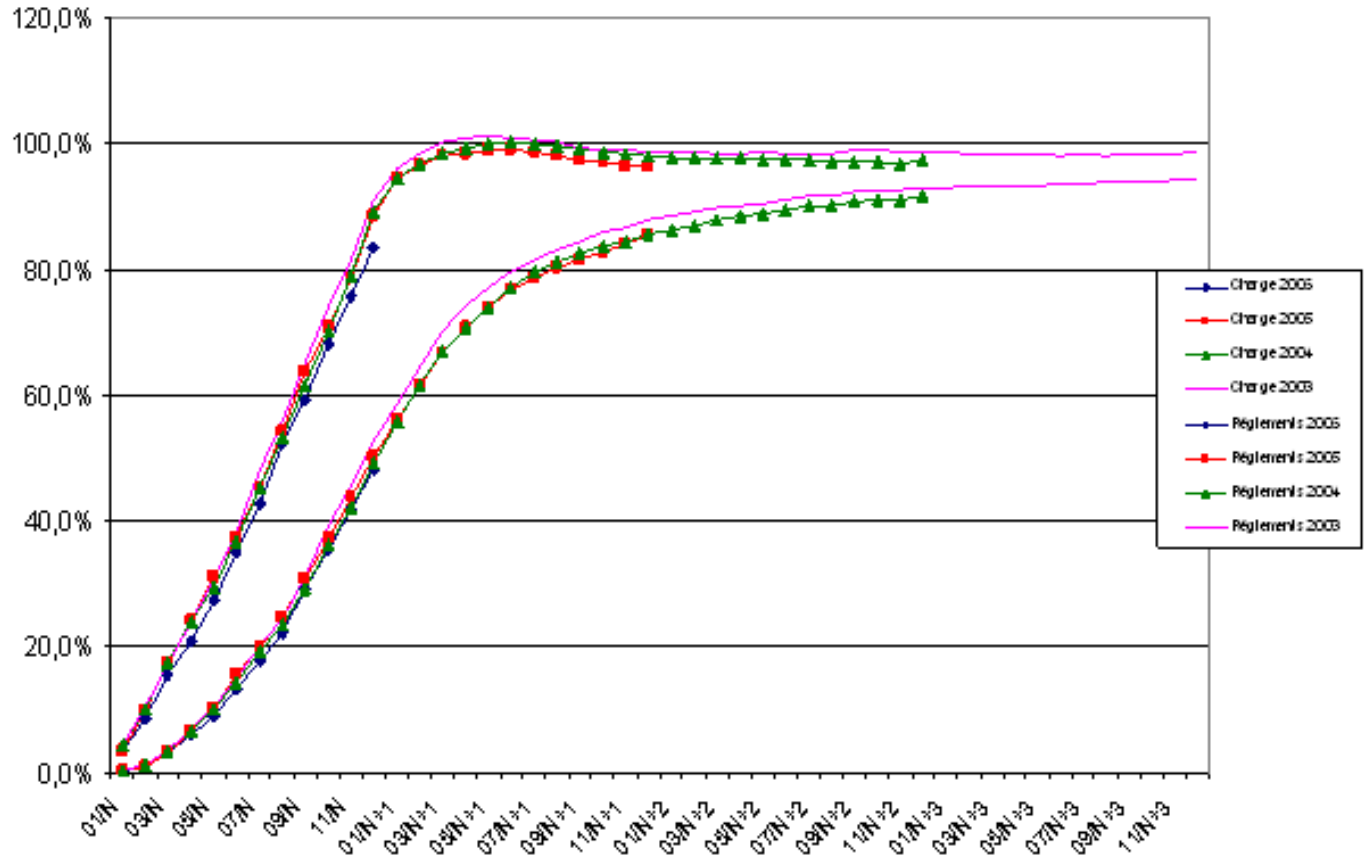

Source : http://www.actuaries.org.uk

# Triangles de Paiements : les méthodes usuelles

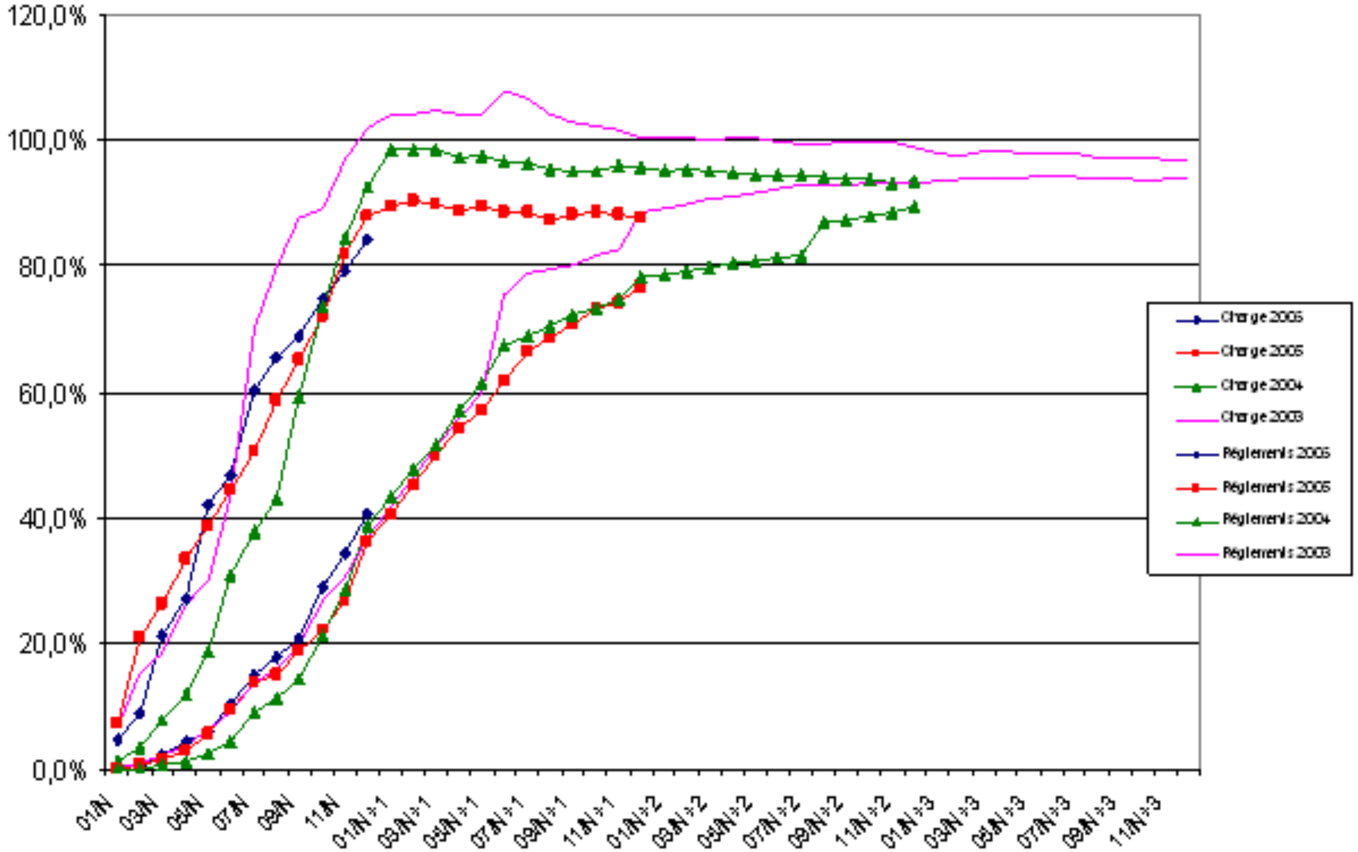



Source : http://www.actuaries.org.uk

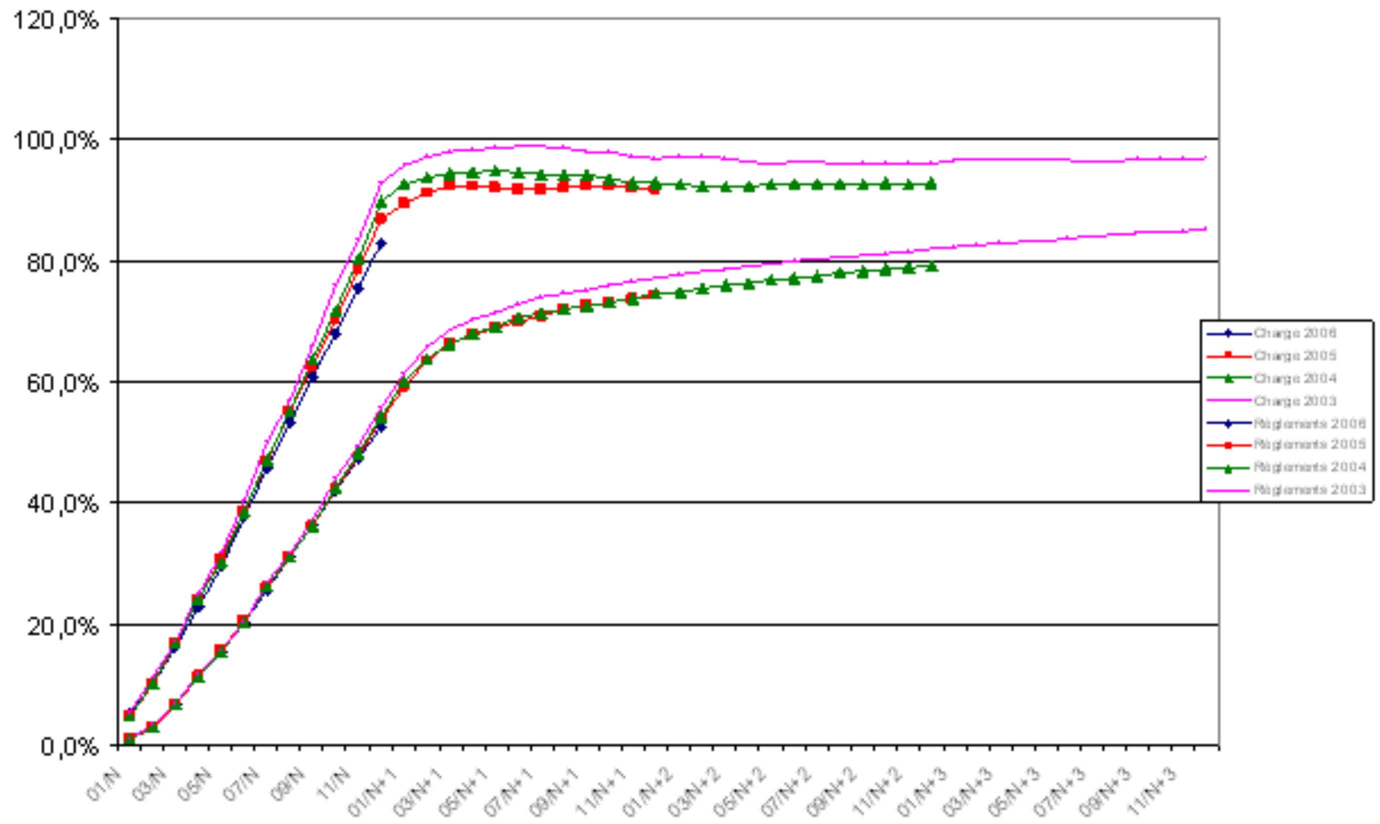
## Exemples de cadence de paiement: multirisques habitation



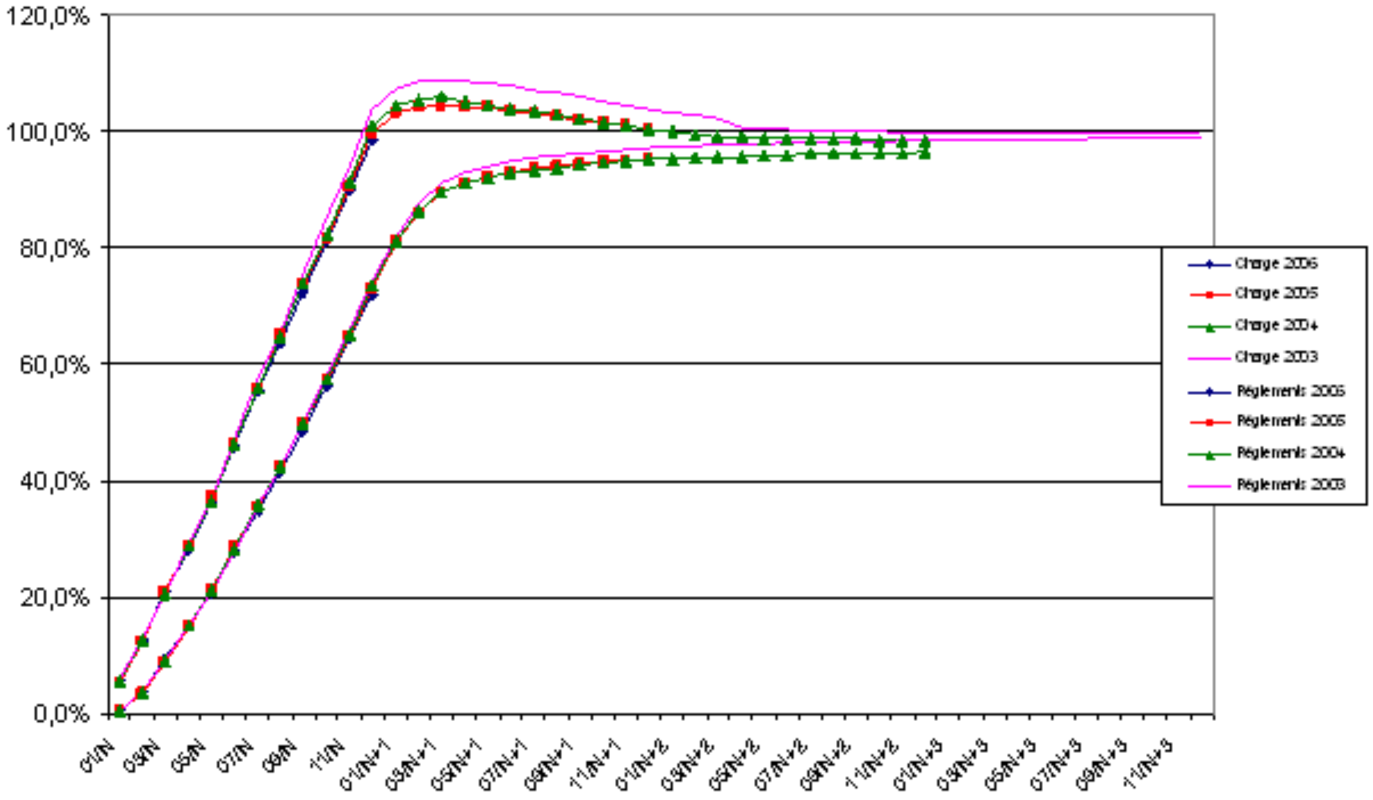
## Exemples de cadence de paiement: risque incendies entreprises



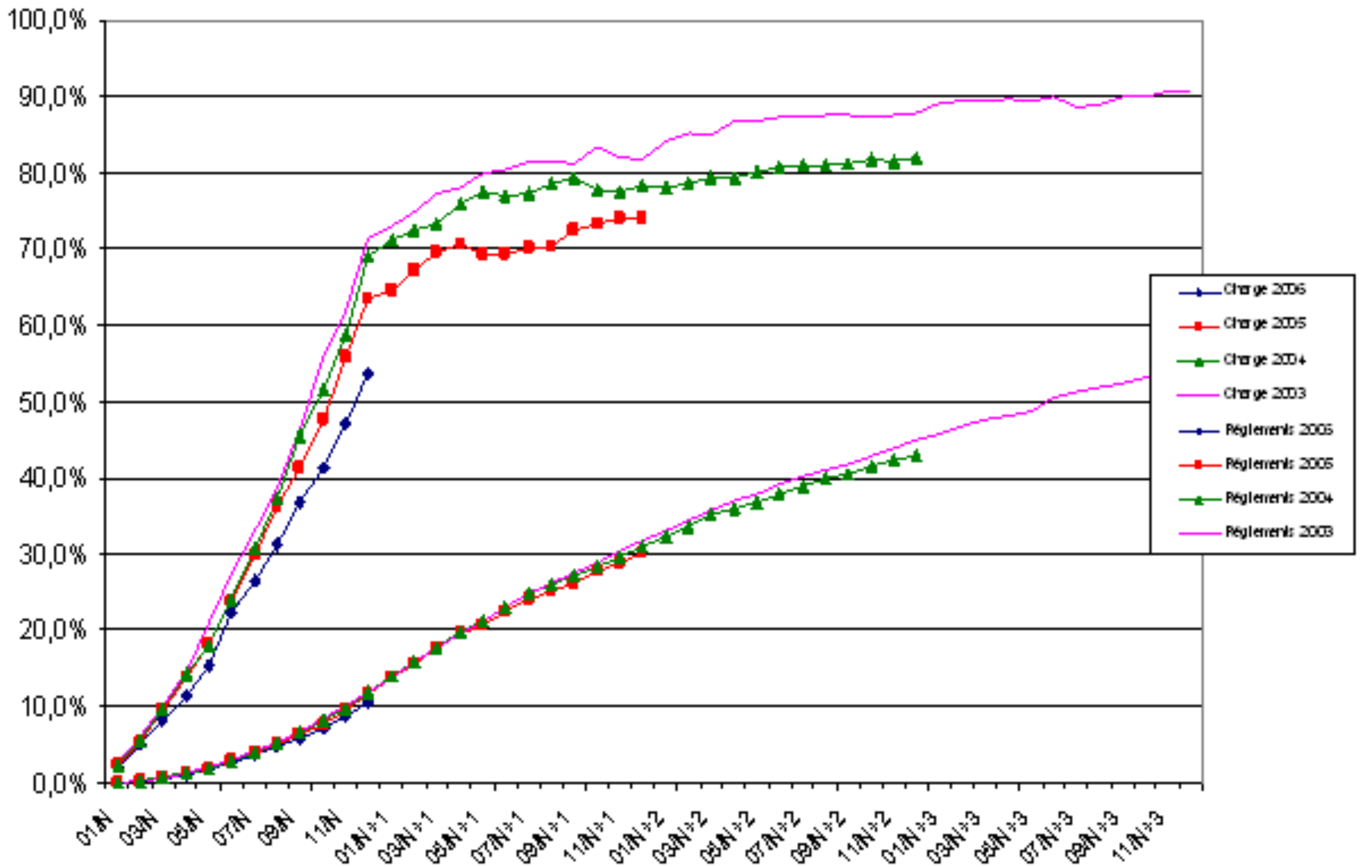
## Exemples de cadence de paiement: automobile (total)



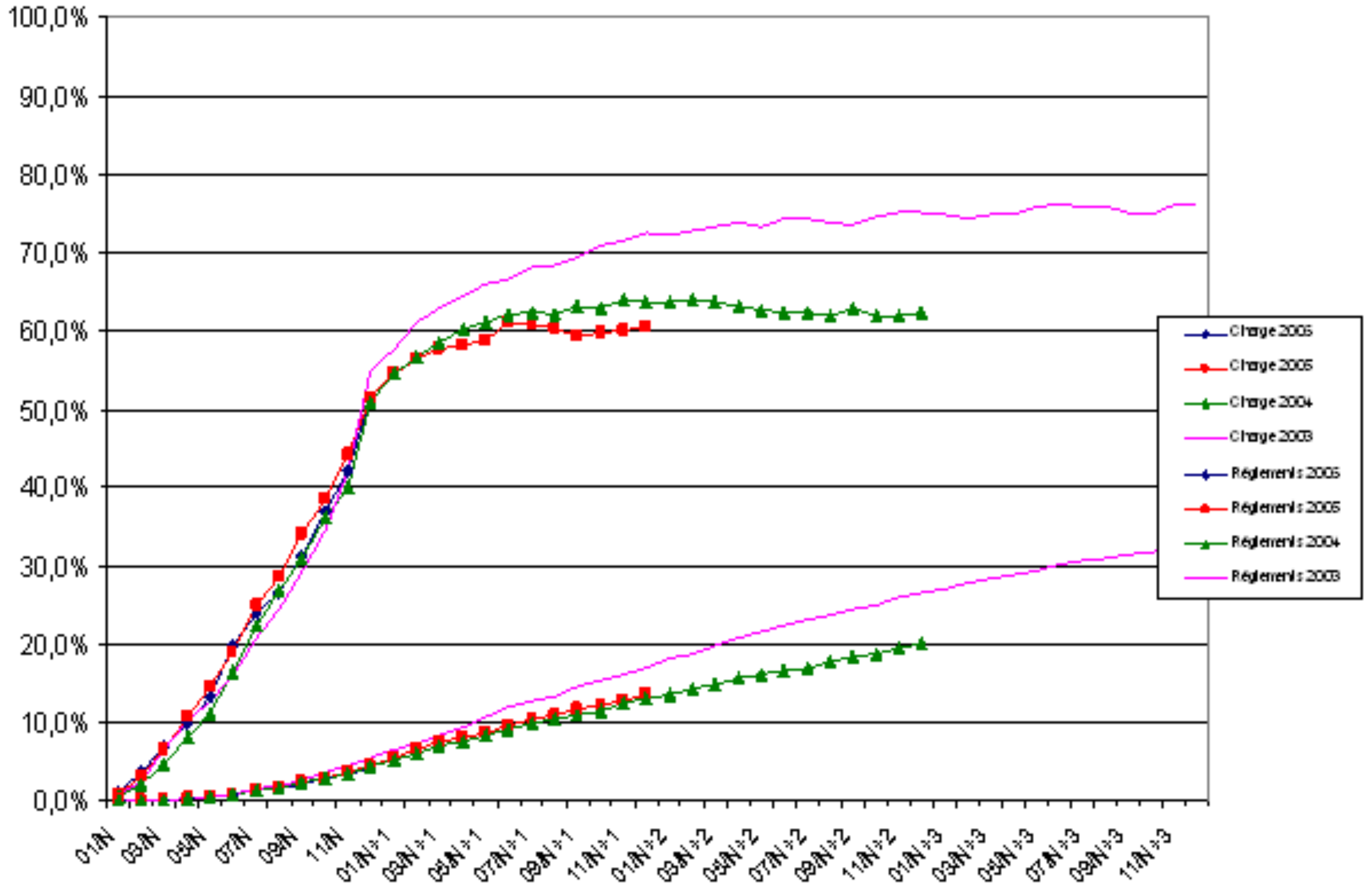
## Exemples de cadence de paiement: automobile matériel



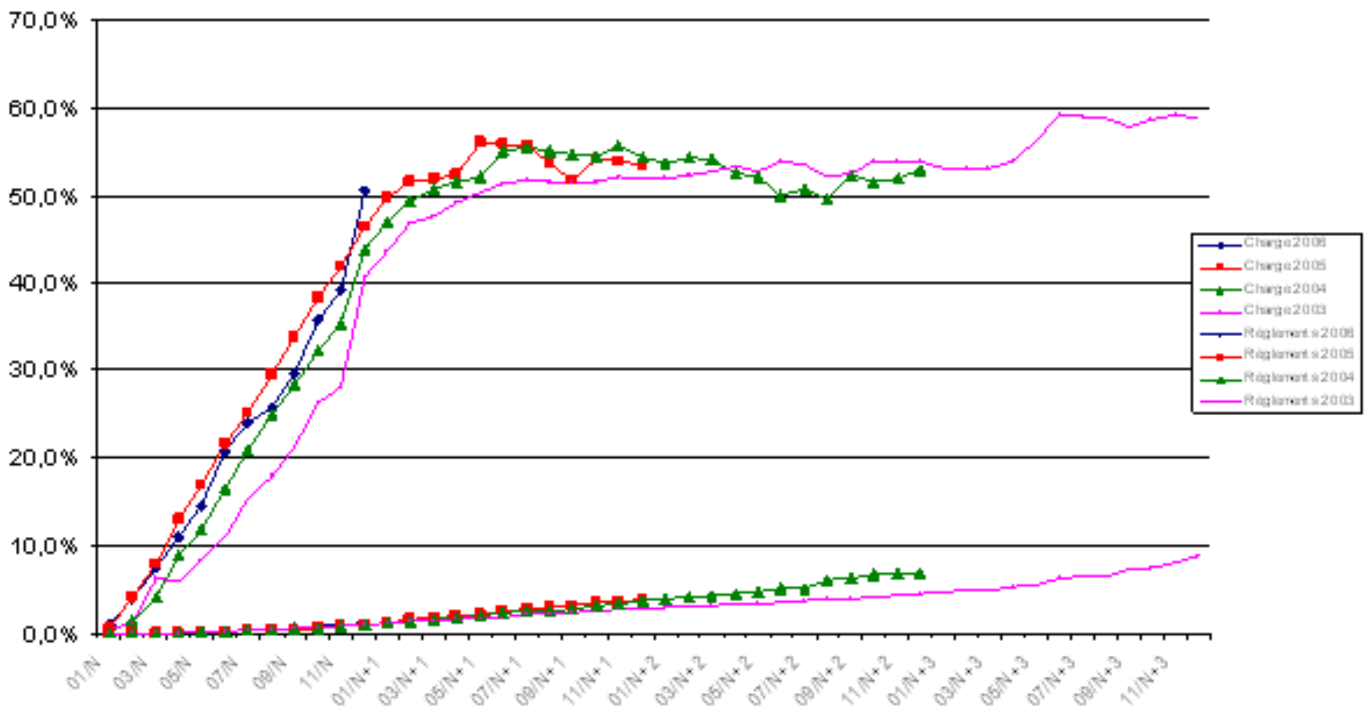
# Exemples de cadence de paiement: automobile corporel



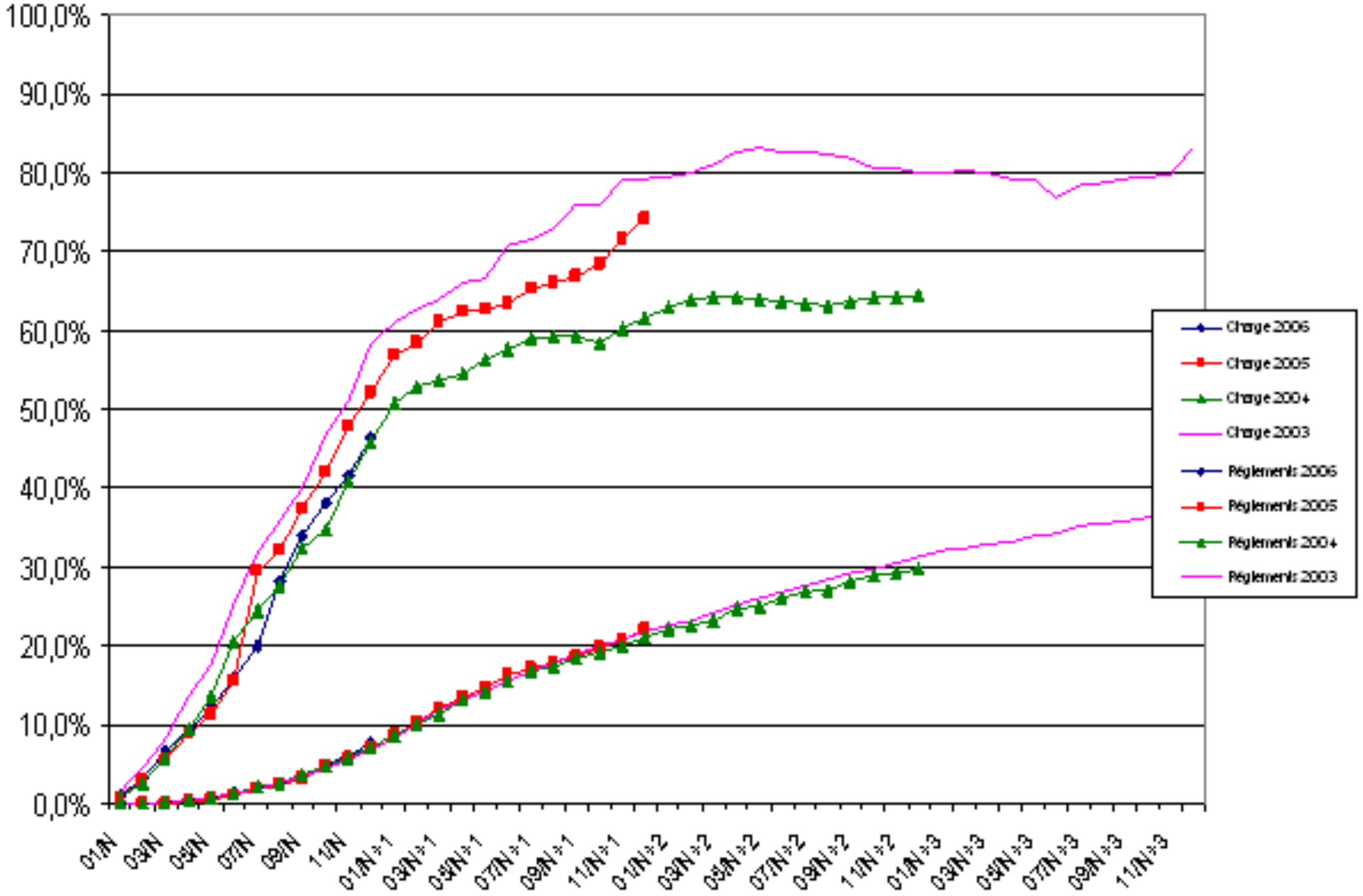
## Exemples de cadence de paiement: responsabilité civile entreprise



# Exemples de cadence de paiement: responabilité civile médicale



# Exemples de cadence de paiement: assurance construction



## Les triangles: incréments de paiements

Notés $Y_{i,j}$, pour l'année de survenance $i$, et l'année de développement $j$,

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 1163 | 39 | 17 | 7 | 21 |
| 1 | 3367 | 1292 | 37 | 24 | 10 | |
| 2 | 3871 | 1474 | 53 | 22 | | |
| 3 | 4239 | 1678 | 103 | | | |
| 4 | 4929 | 1865 | | | | |
| 5 | 5217 | | | | | |

## Les triangles: paiements cumulés

Notés $C_{i,j} = Y_{i,0} + Y_{i,1} + \cdots + Y_{i,j}$, pour l'année de survenance $i$, et l'année de développement $j$,

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | |
| 2 | 3871 | 5345 | 5398 | 5420 | | |
| 3 | 4239 | 5917 | 6020 | | | |
| 4 | 4929 | 6794 | | | | |
| 5 | 5217 | | | | | |

## Les triangles: nombres de sinistres

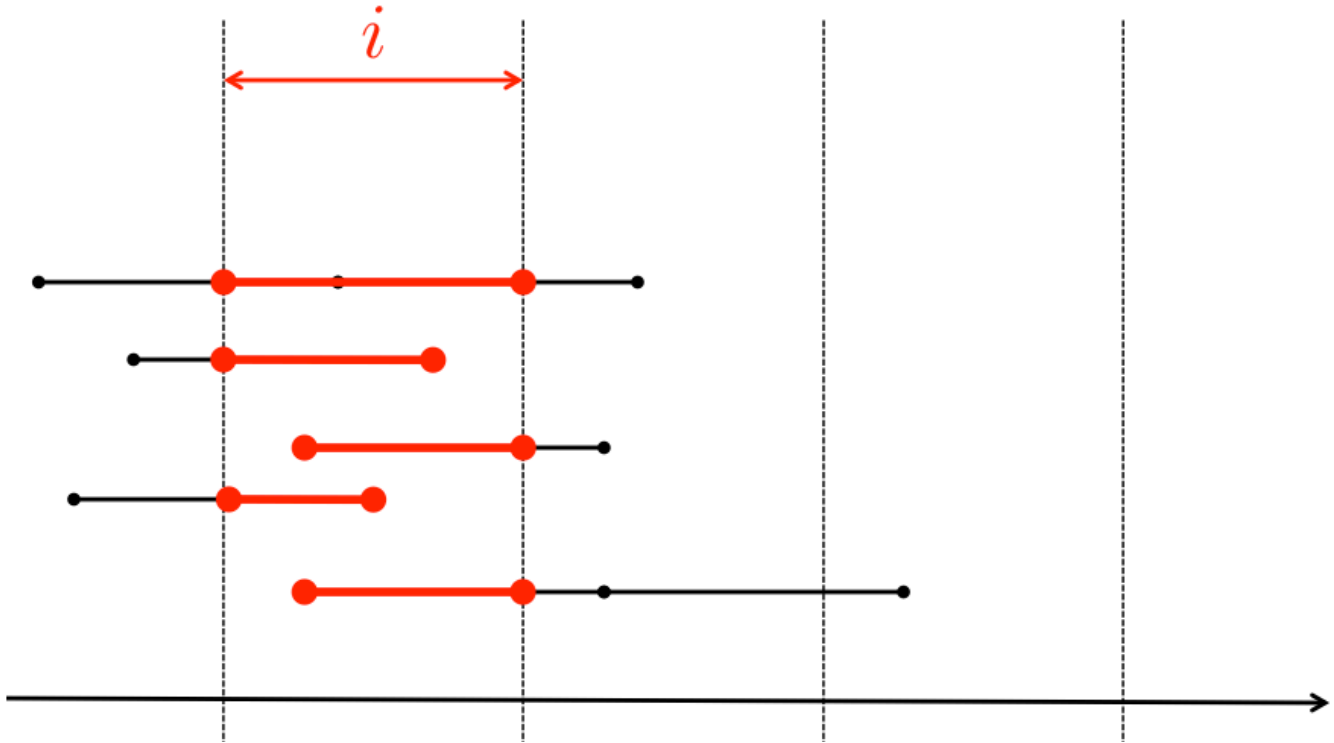
Notés $N_{i,j}$ sinistres survenus l'année $i$ connus (déclarés) au bout de $j$ années,

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1043.4 | 1045.5 | 1047.5 | 1047.7 | 1047.7 | 1047.7 |
| 1 | 1043.0 | 1027.1 | 1028.7 | 1028.9 | 1028.7 | |
| 2 | 965.1 | 967.9 | 967.8 | 970.1 | | |
| 3 | 977.0 | 984.7 | 986.8 | | | |
| 4 | 1099.0 | 1118.5 | | | | |
| 5 | 1076.3 | | | | | |

## La prime acquise

Notée $\pi_i$, prime acquise pour l'année $i$

| année $i$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P_i$ | 4591 | 4672 | 4863 | 5175 | 5673 | 6431 |



## Triangles

```
> rm(list=ls())
> source("http://freakonometrics.free.fr/bases.R")
> ls()
[1] "INCURRED" "NUMBER"   "PAID"     "PREMIUM"
> PAID
     [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 3209 4372 4411 4428 4435 4456
[2,] 3367 4659 4696 4720 4730   NA
[3,] 3871 5345 5398 5420   NA   NA
[4,] 4239 5917 6020   NA   NA   NA
[5,] 4929 6794   NA   NA   NA   NA
[6,] 5217   NA   NA   NA   NA   NA
```

## Triangles ?

Actually, there might be two different cases in practice, the first one being when initial data are missing

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | ○ | ○ | ○ | ● | ● | ● |
| 1 | ○ | ○ | ● | ● | ● | |
| 2 | ○ | ● | ● | ● | | |
| 3 | ● | ● | ● | | | |
| 4 | ● | ● | | | | |
| 5 | ● | | | | | |

In that case it is mainly an index-issue in calculation.

## Triangles ?

Actually, there might be two different cases in practice, the first one being when final data are missing, i.e. some tail factor should be included

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | ● | ● | ● | ● | ○ | ○ |
| 1 | ● | ● | ● | ○ | ○ | ○ |
| 2 | ● | ● | ○ | ○ | ○ | ○ |
| 3 | ● | ○ | ○ | ○ | ○ | ○ |

In that case it is necessary to extrapolate (with past information) the final loss (tail factor).

## The Chain Ladder estimate

We assume here that

$$C_{i,j+1} = \lambda_j \cdot C_{i,j} \text{ for all } i, j = 0, 1, \cdots, n.$$

A natural estimator for $\lambda_j$ based on past history is

$$\widehat{\lambda}_j = \frac{\displaystyle\sum_{i=0}^{n-j} C_{i,j+1}}{\displaystyle\sum_{i=0}^{n-j} C_{i,j}} \text{ for all } j = 0, 1, \cdots, n-1.$$

Hence, it becomes possible to estimate future payments using

$$\widehat{C}_{i,j} = \left[\widehat{\lambda}_{n+1-i} \cdots \widehat{\lambda}_{j-1}\right] C_{i,n+1-i}.$$

## La méthode Chain Ladder, en pratique

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | |
| 2 | 3871 | 5345 | 5398 | 5420 | | |
| 3 | 4239 | 5917 | 6020 | | | |
| 4 | 4929 | 6794 | | | | |
| 5 | 5217 | | | | | |

## La méthode Chain Ladder, en pratique

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | |
| 2 | 3871 | 5345 | 5398 | 5420 | | |
| 3 | 4239 | 5917 | 6020 | | | |
| 4 | 4929 | 6794 | | | | |
| 5 | 5217 | | | | | |

$$\lambda_0 = \frac{4372 + \cdots + 6794}{3209 + \cdots + 4929} \sim 1.38093$$

## La méthode Chain Ladder, en pratique

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | |
| 2 | 3871 | 5345 | 5398 | 5420 | | |
| 3 | 4239 | 5917 | 6020 | | | |
| 4 | 4929 | 6794 | | | | |
| 5 | 5217 | 7204.3 | | | | |

$$\lambda_0 = \frac{4372 + \cdots + 6794}{3209 + \cdots + 4929} \sim 1.38093$$

## La méthode Chain Ladder, en pratique

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | |
| 2 | 3871 | 5345 | 5398 | 5420 | | |
| 3 | 4239 | 5917 | 6020 | | | |
| 4 | 4929 | 6794 | | | | |
| 5 | 5217 | 7204.3 | | | | |

$$\lambda_1 = \frac{4411 + \cdots + 6020}{4372 + \cdots + 5917} \sim 1.01143$$

## La méthode Chain Ladder, en pratique

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | |
| 2 | 3871 | 5345 | 5398 | 5420 | | |
| 3 | 4239 | 5917 | 6020 | | | |
| 4 | 4929 | 6794 | 6871.7 | | | |
| 5 | 5217 | 7204.3 | 7286.7 | | | |

$$\lambda_1 = \frac{4411 + \cdots + 6020}{4372 + \cdots + 5917} \sim 1.01143$$

## La méthode Chain Ladder, en pratique

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | |
| 2 | 3871 | 5345 | 5398 | 5420 | | |
| 3 | 4239 | 5917 | 6020 | | | |
| 4 | 4929 | 6794 | 6871.7 | | | |
| 5 | 5217 | 7204.3 | 7286.7 | | | |

$$\lambda_2 = \frac{4428 + \cdots + 5420}{4411 + \cdots + 5398} \sim 1.00434$$

## La méthode Chain Ladder, en pratique

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | |
| 2 | 3871 | 5345 | 5398 | 5420 | | |
| 3 | 4239 | 5917 | 6020 | 6046.1 | | |
| 4 | 4929 | 6794 | 6871.7 | 6901.5 | | |
| 5 | 5217 | 7204.3 | 7286.7 | 7318.3 | | |

$$\lambda_2 = \frac{4428 + \cdots + 5420}{4411 + \cdots + 5398} \sim 1.00434$$

## La méthode Chain Ladder, en pratique

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | |
| 2 | 3871 | 5345 | 5398 | 5420 | 5430.1 | |
| 3 | 4239 | 5917 | 6020 | 6046.1 | 6057.4 | |
| 4 | 4929 | 6794 | 6871.7 | 6901.5 | 6914.3 | |
| 5 | 5217 | 7204.3 | 7286.7 | 7318.3 | 7331.9 | |

$$\lambda_3 = \frac{4435 + 4730}{4428 + 4720} \sim 1.00186$$

## La méthode Chain Ladder, en pratique

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | 4752.4 |
| 2 | 3871 | 5345 | 5398 | 5420 | 5430.1 | 5455.8 |
| 3 | 4239 | 5917 | 6020 | 6046.1 | 6057.4 | 6086.1 |
| 4 | 4929 | 6794 | 6871.7 | 6901.5 | 6914.3 | 6947.1 |
| 5 | 5217 | 7204.3 | 7286.7 | 7318.3 | 7331.9 | 7366.7 |

$$\lambda_4 = \frac{4456}{4435} \sim 1.00474$$

## La méthode Chain Ladder, en pratique

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | 4752.4 |
| 2 | 3871 | 5345 | 5398 | 5420 | 5430.1 | 5455.8 |
| 3 | 4239 | 5917 | 6020 | 6046.15 | 6057.4 | 6086.1 |
| 4 | 4929 | 6794 | 6871.7 | 6901.5 | 6914.3 | 6947.1 |
| 5 | 5217 | 7204.3 | 7286.7 | 7318.3 | 7331.9 | 7366.7 |

One the triangle has been completed, we obtain the amount of reserves, with respectively 22, 36, 66, 153 and 2150 per accident year, i.e. the total is 2427.

## Computational Issues

```
> library(ChainLadder)
> MackChainLadder(PAID)
MackChainLadder(Triangle = PAID)

  Latest Dev.To.Date Ultimate    IBNR Mack.S.E CV(IBNR)
1  4,456       1.000    4,456     0.0    0.000      NaN
2  4,730       0.995    4,752    22.4    0.639   0.0285
3  5,420       0.993    5,456    35.8    2.503   0.0699
4  6,020       0.989    6,086    66.1    5.046   0.0764
5  6,794       0.978    6,947   153.1   31.332   0.2047
6  5,217       0.708    7,367 2,149.7   68.449   0.0318

             Totals
Latest:   32,637.00
Dev:           0.93
Ultimate: 35,063.99
IBNR:      2,426.99
```

```
Mack.S.E       79.30
CV(IBNR):       0.03
```

## Three ways to look at triangles

There are basically three kind of approaches to model development

- developments as percentages of total incured, i.e. consider $\boldsymbol{\varphi} = (\varphi_0, \varphi_1, \cdots, \varphi_n)$, with $\varphi_0 + \varphi_1 + \cdots + \varphi_n = 1$, such that

$$\mathbb{E}(Y_{i,j}) = \varphi_j \mathbb{E}(C_{i,n}), \text{ where } j = 0, 1, \cdots, n.$$

- developments as rates of total incured, i.e. consider $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \cdots, \gamma_n)$, such that

$$\mathbb{E}(C_{i,j}) = \gamma_j \mathbb{E}(C_{i,n}), \text{ where } j = 0, 1, \cdots, n.$$

- developments as factors of previous estimation, i.e. consider $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \cdots, \lambda_n)$, such that

$$\mathbb{E}(C_{i,j+1}) = \lambda_j \mathbb{E}(C_{i,j}), \text{ where } j = 0, 1, \cdots, n.$$

## Three ways to look at triangles

From a mathematical point of view, it is strictly equivalent to study one of those. Hence,

$$\gamma_j = \varphi_0 + \varphi_1 + \cdots + \varphi_j = \frac{1}{\lambda_j}\frac{1}{\lambda_{j+1}}\cdots\frac{1}{\lambda_{n-1}},$$

$$\lambda_j = \frac{\gamma_{j+1}}{\gamma_j} = \frac{\varphi_0 + \varphi_1 + \cdots + \varphi_j + \varphi_{j+1}}{\varphi_0 + \varphi_1 + \cdots + \varphi_j}$$

$$\varphi_j = \begin{cases} \gamma_0 \text{ if } j = 0 \\ \gamma_j - \gamma_{j-1} \text{ if } j \geq 1 \end{cases} = \begin{cases} \frac{1}{\lambda_0}\frac{1}{\lambda_1}\cdots\frac{1}{\lambda_{n-1}}, \text{ if } j = 0 \\ \frac{1}{\lambda_{j+1}}\frac{1}{\lambda_{j+2}}\cdots\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_j}\frac{1}{\lambda_{j+1}}\cdots\frac{1}{\lambda_{n-1}}, \text{ if } j \geq 1 \end{cases}$$

## Three ways to look at triangles

On the previous triangle,

| | 0 | 1 | 2 | 3 | 4 | $n$ |
|---|---|---|---|---|---|---|
| $\lambda_j$ | 1,38093 | 1,01143 | 1,00434 | 1,00186 | 1,00474 | 1,0000 |
| $\gamma_j$ | 70,819% | 97,796% | 98,914% | 99,344% | 99,529% | 100,000% |
| $\varphi_j$ | 70,819% | 26,977% | 1,118% | 0,430% | 0,185% | 0,000% |

## $d$-triangles

It is possible to define the $d$-triangles, with empirical $\lambda$'s, i.e. $\lambda_{i,j}$

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1.362 | 1.009 | 1.004 | 1.002 | 1.005 | |
| 1 | 1.384 | 1.008 | 1.005 | 1.002 | | |
| 2 | 1.381 | 1.010 | 1.001 | | | |
| 3 | 1.396 | 1.017 | | | | |
| 4 | 1.378 | | | | | |
| 5 | | | | | | |

## The Chain-Ladder estimate

The Chain-Ladder estimate is probably the most popular technique to estimate claim reserves. Let $\mathcal{F}_t$ denote the information avalable at time $t$, or more formally the filtration generated by $\{C_{i,j}, i+j \leq t\}$ - or equivalently $\{X_{i,j}, i+j \leq t\}$

Assume that incremental payments are independent by occurence years, i.e.

$$C_{i_1,\cdot} \text{ and } C_{i_2,\cdot} \text{ are independent for any } i_1 \text{ and } i_2 \qquad [H_1]$$

.

Further, assume that $(C_{i,j})_{j\geq 0}$ is Markov, and more precisely, there exist $\lambda_j$'s and $\sigma_j^2$'s such that

$$\begin{cases} \mathbb{E}(C_{i,j+1}|\mathcal{F}_{i+j}) = \mathbb{E}(C_{i,j+1}|C_{i,j}) = \lambda_j \cdot C_{i,j} & [H_2] \\ \text{Var}(C_{i,j+1}|\mathcal{F}_{i+j}) = \text{Var}(C_{i,j+1}|C_{i,j}) = \sigma_j^2 \cdot C_{i,j} & [H_3] \end{cases}$$

Under those assumption (see Mack (1993)), one gets

$$\mathbb{E}(C_{i,j+k}|\mathcal{F}_{i+j}) = (C_{i,j+k}|C_{i,j}) = \lambda_j \cdot \lambda_{j+1} \cdots \lambda_{j+k-1} C_{i,j}$$

## Testing assumptions

Assumption $H_2$ can be interpreted as a linear regression model, i.e. $Y_i = \beta_0 + X_i \cdot \beta_1 + \varepsilon_i$, $i = 1, \cdots, n$, where $\varepsilon$ is some error term, such that $\mathbb{E}(\varepsilon) = 0$, where $\beta_0 = 0$, $Y_i = C_{i,j+1}$ for some $j$, $X_i = C_{i,j}$, and $\beta_1 = \lambda_j$.

Weighted least squares can be considered, i.e. $\min\left\{\sum_{i=1}^{n-j} \omega_i \left(Y_i - \beta_0 - \beta_1 X_i\right)^2\right\}$ where the $\omega_i$'s are proportional to $\text{Var}(Y_i)^{-1}$. This leads to

$$\min\left\{\sum_{i=1}^{n-j} \frac{1}{C_{i,j}} \left(C_{i,j+1} - \lambda_j C_{i,j}\right)^2\right\}.$$

As in any linear regression model, it is possible to test assumptions $H_1$ and $H_2$, the following graphs can be considered, given $j$

- plot $C_{i,j+1}$'s versus $C_{i,j}$'s. Points should be on the straight line with slope $\widehat{\lambda}_j$.
- plot (standardized) residuals $\varepsilon_{i,j} = \dfrac{C_{i,j+1} - \widehat{\lambda}_j C_{i,j}}{\sqrt{C_{i,j}}}$ versus $C_{i,j}$'s.

## Testing assumptions

$H_1$ is the accident year independent assumption. More precisely, we assume there is no calendar effect.

Define the diagonal $B_k = \{C_{k,0}, C_{k-1,1}, C_{k-2,2} \cdots, C_{2,k-2}, C_{1,k-1}, C_{0,k}\}$. If there is a calendar effect, it should affect adjacent factor lines,

$$A_k = \left\{\frac{C_{k,1}}{C_{k,0}}, \frac{C_{k-1,2}}{C_{k-1,1}}, \frac{C_{k-2,3}}{C_{k-2,2}}, \cdots, \frac{C_{1,k}}{C_{1,k-1}}, \frac{C_{0,k+1}}{C_{0,k}}\right\} = \text{"}\frac{\delta_{k+1}}{\delta_k}\text{"},$$

and

$$A_{k-1} = \left\{\frac{C_{k-1,1}}{C_{k-1,0}}, \frac{C_{k-2,2}}{C_{k-2,1}}, \frac{C_{k-3,3}}{C_{k-3,2}}, \cdots, \frac{C_{1,k-1}}{C_{1,k-2}}, \frac{C_{0,k}}{C_{0,k-1}}\right\} = \text{"}\frac{\delta_k}{\delta_{k-1}}\text{"}.$$

For each $k$, let $N_k^+$ denote the number of elements exceeding the median, and $N_k^-$ the number of elements lower than the mean. The two years are independent, $N_k^+$ and $N_k^-$ should be "closed", i.e. $N_k = \min\left(N_k^+, N_k^-\right)$ should be "closed" to $\left(N_k^+ + N_k^-\right)/2$.

## Testing assumptions

Since $N_k^-$ and $N_k^+$ are two binomial distributions $\mathcal{B}\left(p=1/2, n=N_k^-+N_k^+\right)$, then

$$\mathbb{E}\left(N_k\right)=\frac{n_k}{2}-\left(\begin{array}{c} n_k-1 \\ m_k \end{array}\right)\frac{n_k}{2^{n_k}} \text{ where } n_k=N_k^++N_k^- \text{ and } m_k=\left[\frac{n_k-1}{2}\right]$$

and

$$V\left(N_k\right)=\frac{n_k\left(n_k-1\right)}{2}-\left(\begin{array}{c} n_k-1 \\ m_k \end{array}\right)\frac{n_k\left(n_k-1\right)}{2^{n_k}}+\mathbb{E}\left(N_k\right)-\mathbb{E}\left(N_k\right)^2.$$

Under some normality assumption on $N$, a 95% confidence interval can be derived, i.e. $\mathbb{E}(Z) \pm 1.96\sqrt{V(Z)}$.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0.734 | .0991 | 0.996 | 0.998 | 0.995 | |
| 1 | 0.723 | 0.992 | 0.995 | 0.998 | | |
| 2 | 0.724 | 0.990 | 0.996 | | | |
| 3 | 0.716 | 0.983 | | | | |
| 4 | 0.725 | | | | | |
| 5 | | | | | | |
| 6 | 0.724 | 0.991 | 0.996 | 0.998 | 0.995 | |

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | + | + | + | + | · | |
| 1 | − | + | − | − | | |
| 2 | · | − | · | | | |
| 3 | − | − | | | | |
| 4 | + | | | | | |
| 5 | | | | | | |

## From Chain-Ladder to Grossing-Up

The idea of the Chain-Ladder technique was to estimate the $\lambda_j$'s, so that we can derive estimates for $C_{i,n}$, since

$$\widehat{C}_{i,n} = \widehat{C}_{i,n-i} \cdot \prod_{k=n-i+1}^{n} \widehat{\lambda}_k$$

Based on the Chain-Ladder link ratios, $\widehat{\lambda}$, it is possible to define grossing-up coefficients

$$\widehat{\gamma}_j = \prod_{k=j}^{n} \frac{1}{\widehat{\lambda}_k}$$

and thus, the total loss incured for accident year $i$ is then

$$\widehat{C}_{i,n} = \widehat{C}_{i,n-i} \cdot \frac{\widehat{\gamma}_n}{\widehat{\gamma}_{n-i}}$$

## Variant of the Chain-Ladder Method (1)

Historically (see e.g.), the *natural* idea was to consider a (standard) average of individual link ratios.

Several techniques have been introduces to study individual link-ratios.

A first idea is to consider a simple linear model, $\lambda_{i,j} = a_j i + b_j$. Using OLS techniques, it is possible to estimate those coefficients simply. Then, we project those ratios using predicted one, $\widehat{\lambda}_{i,j} = \widehat{a}_j i + \widehat{b}_j$.

## Variant of the Chain-Ladder Method (2)

A second idea is to assume that $\lambda_j$ is the weighted sum of $\lambda_{\cdots,j}$'s,

$$\widehat{\lambda}_j = \frac{\displaystyle\sum_{i=0}^{j-1} \omega_{i,j}\lambda_{i,j}}{\displaystyle\sum_{i=0}^{j-1} \omega_{i,j}}$$

If $\omega_{i,j} = C_{i,j}$ we obtain the chain ladder estimate. An alternative is to assume that $\omega_{i,j} = i + j + 1$ (in order to give more weight to recent years).

## Variant of the Chain-Ladder Method (3)

Here, we assume that cumulated run-off triangles have an exponential trend, i.e.

$$C_{i,j} = \alpha_j \exp(i \cdot \beta_j).$$

In order to estimate the $\alpha_j$'s and $\beta_j$'s is to consider a linear model on $\log C_{i,j}$,

$$\log C_{i,j} = \underbrace{a_j}_{\log(\alpha_j)} + \beta_j \cdot i + \varepsilon_{i,j}.$$

Once the $\beta_j$'s have been estimated, set $\widehat{\gamma}_j = \exp(\widehat{\beta}_j)$, and define

$$\Gamma_{i,j} = \widehat{\gamma}_j^{n-i-j} \cdot C_{i,j}.$$

## The extended link ratio family of estimators

For convenience, link ratios are factors that give relation between cumulative payments of one development year (say $j$) and the next development year $(j+1)$. They are simply the ratios $y_i/x_i$, where $x_i$'s are cumulative payments year $j$ (i.e. $x_i = C_{i,j}$) and $y_i$'s are cumulative payments year $j+1$ (i.e. $y_i = C_{i,j+1}$).

For example, the Chain Ladder estimate is obtained as

$$\widehat{\lambda}_j = \frac{\sum_{i=0}^{n-j} y_i}{\sum_{k=0}^{n-j} x_k} = \sum_{i=0}^{n-j} \frac{x_i}{\sum_{k=1}^{n-j} x_k} \cdot \frac{y_i}{x_i}.$$

But several other link ratio techniques can be considered, e.g.

$$\widehat{\lambda}_j = \frac{1}{n-j+1} \sum_{i=0}^{n-j} \frac{y_i}{x_i}, \text{ i.e. the simple arithmetic mean,}$$

$$\widehat{\lambda}_j = \left( \prod_{i=0}^{n-j} \frac{y_i}{x_i} \right)^{n-j+1}, \text{ i.e. the geometric mean,}$$

$$\widehat{\lambda}_j = \sum_{i=0}^{n-j} \frac{x_i^2}{\sum_{k=1}^{n-j} x_k^2} \cdot \frac{y_i}{x_i},$$ i.e. the weighted average "by volume squared",

Hence, these techniques can be related to weighted least squares, i.e.

$$y_i = \beta x_i + \varepsilon_i, \text{ where } \varepsilon_i \sim \mathcal{N}(0, \sigma^2 x_i^\delta), \text{ for some } \delta > 0.$$

E.g. if $\delta = 0$, we obtain the arithmetic mean, if $\delta = 1$, we obtain the Chain Ladder estimate, and if $\delta = 2$, the weighted average "by volume squared".

The interest of this regression approach, is that standard error for predictions can be derived, under standard (and testable) assumptions. Hence

- standardized residuals $(\sigma x_i^{\delta/2})^{-1}\varepsilon_i$ are $\mathcal{N}(0,1)$, i.e. QQ plot
- $\mathbb{E}(y_i|x_i) = \beta x_i$, i.e. graph of $x_i$ versus $y_i$.

## Properties of the Chain-Ladder estimate

Further

$$\widehat{\lambda}_j = \frac{\sum_{i=0}^{n-j-1} C_{i,j+1}}{\sum_{i=0}^{n-j-1} C_{i,j}}$$

is an unbiased estimator for $\lambda_j$, given $\mathcal{F}_j$, and $\widehat{\lambda}_j$ and $\widehat{\lambda}_{j+h}$ are non-correlated, given $\mathcal{F}_j$. Hence, an unbiased estimator for $\mathbb{E}(C_{i,j}|\mathcal{F}_n)$ is

$$\widehat{C}_{i,j} = \widehat{\lambda}_{n-i} \cdot \widehat{\lambda}_{n-i+1} \cdots \widehat{\lambda}_{j-2} \left(\widehat{\lambda}_{j-1} - 1\right) \cdot C_{i,n-i}.$$

Recall that $\widehat{\lambda}_j$ is the estimator with minimal variance among all linear estimators obtained from $\lambda_{i,j} = C_{i,j+1}/C_{i,j}$'s. Finally, recall that

$$\widehat{\sigma}_j^2 = \frac{1}{n-j-1} \sum_{i=0}^{n-j-1} \left(\frac{C_{i,j+1}}{C_{i,j}} - \widehat{\lambda}_j\right)^2 \cdot C_{i,j}$$

is an unbiased estimator of $\sigma_j^2$, given $\mathcal{F}_j$ (see Mack (1993) or Denuit & Charpentier (2005)).

## Prediction error of the Chain-Ladder estimate

We stress here that estimating reserves is a prediction process: based on past observations, we predict future amounts. Recall that prediction error can be explained as follows,

$$\underbrace{\mathbb{E}[(Y-\widehat{Y})^2]}_{\text{prediction variance}} = \mathbb{E}\Big[\Big((Y-\mathbb{E}Y)+(\mathbb{E}(Y)-\widehat{Y})\Big)^2\Big]$$

$$\approx \underbrace{\mathbb{E}[(Y-\mathbb{E}Y)^2]}_{\text{process variance}}+\underbrace{\mathbb{E}[(\mathbb{E}Y-\widehat{Y})^2]}_{\text{estimation variance}}.$$

- the process variance reflects randomness of the random variable
- the estimation variance reflects uncertainty of statistical estimation

## Process variance of reserves per occurrence year

The amount of reserves for accident year $i$ is simply

$$\widehat{R}_i = \left(\widehat{\lambda}_{n-i} \cdot \widehat{\lambda}_{n-i+1} \cdots \widehat{\lambda}_{n-2}\widehat{\lambda}_{n-1} - 1\right) \cdot C_{i,n-i}.$$

Note that $\mathbb{E}(\widehat{R}_i|\mathcal{F}_n) = R_i$ Since

$$\begin{aligned}\text{Var}(\widehat{R}_i|\mathcal{F}_n) &= \text{Var}(C_{i,n}|\mathcal{F}_n) = \text{Var}(C_{i,n}|C_{i,n-i}) \\ &= \sum_{k=i+1}^{n} \prod_{l=k+1}^{n} \lambda_l^2 \sigma_k^2 \mathbb{E}[C_{i,k}|C_{i,n-i}]\end{aligned}$$

and a natural estimator for this variance is then

$$\begin{aligned}\widehat{\text{Var}}(\widehat{R}_i|\mathcal{F}_n) &= \sum_{k=i+1}^{n} \prod_{l=k+1}^{n} \widehat{\lambda}_l^2 \widehat{\sigma}_k^2 \widehat{C}_{i,k} \\ &= \widehat{C}_{i,n} \sum_{k=i+1}^{n} \frac{\widehat{\sigma}_k^2}{\widehat{\lambda}_k^2 \widehat{C}_{i,k}}.\end{aligned}$$

Note that it is possible to get not only the variance of the ultimate cumulate payments, but also the variance of any increment. Hence

$$\begin{aligned}\text{Var}(Y_{i,j}|\mathcal{F}_n) &= \text{Var}(Y_{i,j}|C_{i,n-i}) \\ &= \mathbb{E}[\text{Var}(Y_{i,j}|C_{i,j-1})|C_{i,n-i}] + \text{Var}[\mathbb{E}(Y_{i,j}|C_{i,j-1})|C_{i,n-i}] \\ &= \mathbb{E}[\sigma_i^2 C_{i,j-1}|C_{i,n-i}] + \text{Var}[(\lambda_{j-1}-1)C_{i,j-1}|C_{i,n-i}]\end{aligned}$$

and a natural estimator for this variance is then

$$\widehat{\text{Var}}(Y_{i,j}|\mathcal{F}_n) = \widehat{\text{Var}}(C_{i,j}|\mathcal{F}_n) + (1-2\widehat{\lambda}_{j-1})\widehat{\text{Var}}(C_{i,j-1}|\mathcal{F}_n)$$

where, from the expressions given above,

$$\widehat{\text{Var}}(C_{i,j}|\mathcal{F}_n) = C_{i,n-i}\sum_{k=i+1}^{j-1}\frac{\widehat{\sigma}_k^2}{\widehat{\lambda}_k^2\widehat{C}_{i,k}}.$$

## Parameter variance when estimating reserves per occurrence year

So far, we have obtained an estimate for the process error of technical risks (increments or cumulated payments). But since parameters $\lambda_j$'s and $\sigma_j^2$ are estimated from past information, there is an additional potential error, also called parameter error (or estimation error). Hence, we have to quantify $\mathbb{E}\left([R_i - \widehat{R}_i]^2\right)$. In order to quantify that error, Murphy (1994) assume the following underlying model,

$$C_{i,j} = \lambda_{j-1} \cdot C_{i,j-1} + \eta_{i,j}$$

with independent variables $\eta_{i,j}$. From the structure of the conditional variance,

$$\text{Var}(C_{i,j+1}|\mathcal{F}_{i+j}) = \text{Var}(C_{i,j+1}|C_{i,j}) = \sigma_j^2 \cdot C_{i,j},$$

## Parameter variance when estimating reserves per occurrence year

it is natural to write the equation above

$$C_{i,j} = \lambda_{j-1}C_{i,j-1} + \sigma_{j-1}\sqrt{C_{i,j-1}}\varepsilon_{i,j},$$

with independent *and centered variables with unit variance* $\varepsilon_{i,j}$. Then

$$\mathbb{E}\left([R_i - \widehat{R}_i]^2|\mathcal{F}_n\right) = \widehat{R}_i^2\left(\sum_{k=0}^{n-i-1}\frac{\widehat{\sigma}_{i+k}^2}{\widehat{\lambda}_{i+k}^2\sum C_{\cdot,i+k}} + \frac{\widehat{\sigma}_{n-1}^2}{[\widehat{\lambda}_{n-1}-1]^2\sum C_{\cdot,i+k}}\right)$$

Based on that estimator, it is possible to derive the following estimator for the Conditional Mean Square Error of reserve prediction for occurrence year $i$,

$$CMSE_i = \widehat{\text{Var}}(\widehat{R}_i|\mathcal{F}_n) + \mathbb{E}\left([R_i - \widehat{R}_i]^2|\mathcal{F}_n\right).$$

## Variance of global reserves (for all occurrence years)

The estimate total amount of reserves is $\widehat{\text{Var}}(\widehat{R}) = \widehat{\text{Var}}(\widehat{R}_1) + \cdots + \widehat{\text{Var}}(\widehat{R}_n)$.

In order to derive the conditional mean square error of reserve prediction, define the covariance term, for $i < j$, as

$$CMSE_{i,j} = \widehat{R}_i \widehat{R}_j \left( \sum_{k=i}^{n} \frac{\widehat{\sigma}_{i+k}^2}{\widehat{\lambda}_{i+k}^2 \sum C_{\cdot,k}} + \frac{\widehat{\sigma}_j^2}{[\widehat{\lambda}_{j-1} - 1]\widehat{\lambda}_{j-1} \sum C_{\cdot,j+k}} \right),$$

then the conditional mean square error of overall reserves

$$CMSE = \sum_{i=1}^{n} CMSE_i + 2 \sum_{j>i} CMSE_{i,j}.$$

## Application on our triangle

```
> MackChainLadder(PAID)

  Latest Dev.To.Date Ultimate     IBNR  Mack.S.E CV(IBNR)
1  4,456       1.000    4,456      0.0     0.000      NaN
2  4,730       0.995    4,752     22.4     0.639   0.0285
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5  6,794       0.978    6,947    153.1    31.332   0.2047
6  5,217       0.708    7,367  2,149.7    68.449   0.0318

                 Totals
Latest:    32,637.00
Dev:            0.93
Ultimate:  35,063.99
IBNR:       2,426.99
Mack.S.E       79.30
CV(IBNR):       0.03
```

## Application on our triangle

```
> MackChainLadder(PAID)$f
[1] 1.380933 1.011433 1.004343 1.001858 1.004735 1.000000
> MackChainLadder(PAID)$f.se
[1] 5.175575e-03 2.248904e-03 3.808886e-04 2.687604e-04 9.710323e-05
> MackChainLadder(PAID)$sigma
[1] 0.724857769 0.320364221 0.045872973 0.025705640 0.006466667
> MackChainLadder(PAID)$sigma^2
[1] 5.254188e-01 1.026332e-01 2.104330e-03 6.607799e-04 4.181778e-05
```

## A short word on Munich Chain Ladder

Munich chain ladder is an extension of Mack's technique based on paid $(P)$ and incurred $(I)$ losses.

Here we adjust the chain-ladder link-ratios $\lambda_j$'s depending if the momentary $(P/I)$ ratio is above or below average. It integrated correlation of residuals between $P$ vs. $I/P$ and $I$ vs. $P/I$ chain-ladder link-ratio to estimate the correction factor.

Use standard Chain Ladder technique on the two triangles.

## A short word on Munich Chain Ladder

The (standard) payment triangle, $P$

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | 4752.4 |
| 2 | 3871 | 5345 | 5398 | 5420 | 5430.1 | 5455.8 |
| 3 | 4239 | 5917 | 6020 | 6046.15 | 6057.4 | 6086.1 |
| 4 | 4929 | 6794 | 6871.7 | 6901.5 | 6914.3 | 6947.1 |
| 5 | 5217 | 7204.3 | 7286.7 | 7318.3 | 7331.9 | 7366.7 |

## Computational Issues

```
> MackChainLadder(PAID)

  Latest Dev.To.Date Ultimate    IBNR Mack.S.E CV(IBNR)
1  4,456       1.000    4,456     0.0    0.000      NaN
2  4,730       0.995    4,752    22.4    0.639   0.0285
3  5,420       0.993    5,456    35.8    2.503   0.0699
4  6,020       0.989    6,086    66.1    5.046   0.0764
5  6,794       0.978    6,947   153.1   31.332   0.2047
6  5,217       0.708    7,367 2,149.7   68.449   0.0318

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Latest:     32,637.00
Dev:             0.93
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IBNR:        2,426.99
Mack.S.E        79.30
CV(IBNR):        0.03
```

## A short word on Munich Chain Ladder

The Incurred Triangle (I) with estimated losses,

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 4795 | 4629 | 4497 | 4470 | 4456 | 4456 |
| 1 | 5135 | 4949 | 4783 | 4760 | 4750 | 4750.0 |
| 2 | 5681 | 5631 | 5492 | 5470 | 5455.8 | 5455.8 |
| 3 | 6272 | 6198 | 6131 | 6101.1 | 6085.3 | 6085.3 |
| 4 | 7326 | 7087 | 6920.1 | 6886.4 | 6868.5 | 6868.5 |
| 5 | 7353 | 7129.1 | 6991.2 | 6927.3 | 6909.3 | 6909.3 |

## Computational Issues

```
> MackChainLadder(INCURRED)

  Latest Dev.To.Date Ultimate   IBNR Mack.S.E CV(IBNR)
1  4,456        1.00    4,456    0.0    0.000      NaN
2  4,750        1.00    4,750    0.0    0.975      Inf
3  5,470        1.00    5,456  -14.2    4.747   -0.334
4  6,131        1.01    6,085  -45.7    8.305   -0.182
5  7,087        1.03    6,869 -218.5   71.443   -0.327
6  7,353        1.06    6,909 -443.7  180.166   -0.406

              Totals
Latest:    35,247.00
Dev:            1.02
Ultimate:  34,524.83
IBNR:        -722.17
Mack.S.E      201.00
```

(keep only here $\widehat{C}_n = 34,524$, to be compared with the previous $35,067$.

## Computational Issues

```
> MunichChainLadder(PAID,INCURRED)

  Latest Paid Latest Incurred Latest P/I Ratio Ult. Paid Ult.
    Incurred Ult. P/I Ratio
1        4,456            4,456            1.000     4,456
    4,456             1
2        4,730            4,750            0.996     4,753
    4,750             1
3        5,420            5,470            0.991     5,455
    5,454             1
4        6,020            6,131            0.982     6,086
    6,085             1
5        6,794            7,087            0.959     6,983
    6,980             1
6        5,217            7,353            0.710     7,538
    7,533             1

```

```
Totals
            Paid  Incurred  P/I Ratio
Latest:    32,637    35,247       0.93
Ultimate:  35,271    35,259       1.00
```

It is possible to get a model mixing the two approaches together...

## Bornhuetter Ferguson

One of the difficulties with using the chain ladder method is that reserve forecasts can be quite unstable. The Bornhuetter & Ferguson (1972) method provides a procedure for stabilizing such estimates.

Recall that in the standard chain ladder model,

$$\widehat{C}_{i,n} = \widehat{F}_n \cdot C_{i,n-i}, \text{ where } \widehat{F}_n = \prod_{k=n-i}^{n-1} \widehat{\lambda}_k$$

If $\widehat{R}_i$ denotes the estimated outstanding reserves,

$$\widehat{R}_i = \widehat{C}_{i,n} - C_{i,n-i} = \widehat{C}_{i,n} \cdot \frac{\widehat{F}_n - 1}{\widehat{F}_n}.$$

## Bornhuetter Ferguson

For a bayesian interpretation of the Bornhutter-Ferguson model, England & Verrall (2002) considered the case where incremental paiments $Y_{i,j}$ are i.i.d. overdispersed Poisson variables. Here

$$\mathbb{E}(Y_{i,j}) = a_i b_j \text{ and } \text{Var}(Y_{i,j}) = \varphi \cdot a_i b_j,$$

where we assume that $b_1 + \cdots + b_n = 1$. Parameter $a_i$ is assumed to be a drawing of a random variable $A_i \sim \mathcal{G}(\alpha_i, \beta_i)$, so that $\mathbb{E}(A_i) = \alpha_i / \beta_i$, so that

$$\mathbb{E}(C_{i,n}) = \frac{\alpha_i}{\beta_i} = C_i^\star,$$

which is simply a prior expectation of the final loss amount.

## Bornhuetter Ferguson

The posterior distribution of $X_{i,j+1}$ is then

$$\mathbb{E}(X_{i,j+1}|\mathcal{F}_{i+j}) = \left( Z_{i,j+1}C_{i,j} + [1 - Z_{i,j+1}]\frac{C_i^\star}{\widehat{F}_j} \right) \cdot (\lambda_j - 1)$$

where $Z_{i,j+1} = \dfrac{\widehat{F}_j^{-1}}{\beta\varphi + \widehat{F}_j}$, where $\widehat{F}_j = \lambda_{j+1} \cdots \lambda_n$.

Hence, Bornhutter-Ferguson technique can be interpreted as a Bayesian method, and a a credibility estimator (since bayesian with conjugated distributed leads to credibility).

## Bornhuetter Ferguson

The underlying assumptions are here

- assume that accident years are independent
- assume that there exist parameters $\boldsymbol{\mu} = (\mu_0, \cdots, \mu_n)$ and a pattern $\boldsymbol{\beta} = (\beta_0, \beta_1, \cdots, \beta_n)$ with $\beta_n = 1$ such that

$$\mathbb{E}(C_{i,0}) = \beta_0 \mu_i$$

$$\mathbb{E}(C_{i,j+k}|\mathcal{F}_{i+j}) = C_{i,j} + [\beta_{j+k} - \beta_j] \cdot \mu_i$$

Hence, one gets that $\mathbb{E}(C_{i,j}) = \beta_j \mu_i$.

The sequence $(\beta_j)$ denotes the claims development pattern. The Bornhuetter-Ferguson estimator for $\mathbb{E}(C_{i,n}|Ci, 1, \cdots, C_{i,j})$ is

$$\widehat{C}_{i,n} = C_{i,j} + [1 - \widehat{\beta}_{j-i}]\widehat{\mu}_i$$

where $\widehat{\mu}_i$ is an estimator for $\mathbb{E}(C_{i,n})$.

If we want to relate that model to the classical Chain Ladder one,

$$\beta_j \text{ is } \prod_{k=j+1}^{n} \frac{1}{\lambda_k}$$

Consider the classical triangle. Assume that the estimator $\widehat{\mu}_i$ is a plan value (obtain from some business plan). For instance, consider a 105% loss ratio per accident year.

## Bornhuetter Ferguson

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| premium | 4591 | 4692 | 4863 | 5175 | 5673 | 6431 |
| $\widehat{\mu}_i$ | 4821 | 4927 | 5106 | 5434 | 5957 | 6753 |
| $\lambda_i$ | $1,380$ | $1,011$ | $1,004$ | $1,002$ | $1,005$ | |
| $\beta_i$ | $0,708$ | $0,978$ | $0,989$ | $0,993$ | $0,995$ | |
| $\widehat{C}_{i,n}$ | 4456 | 4753 | 5453 | 6079 | 6925 | 7187 |
| $\widehat{R}_i$ | 0 | 23 | 33 | 59 | 131 | 1970 |

## Boni-Mali

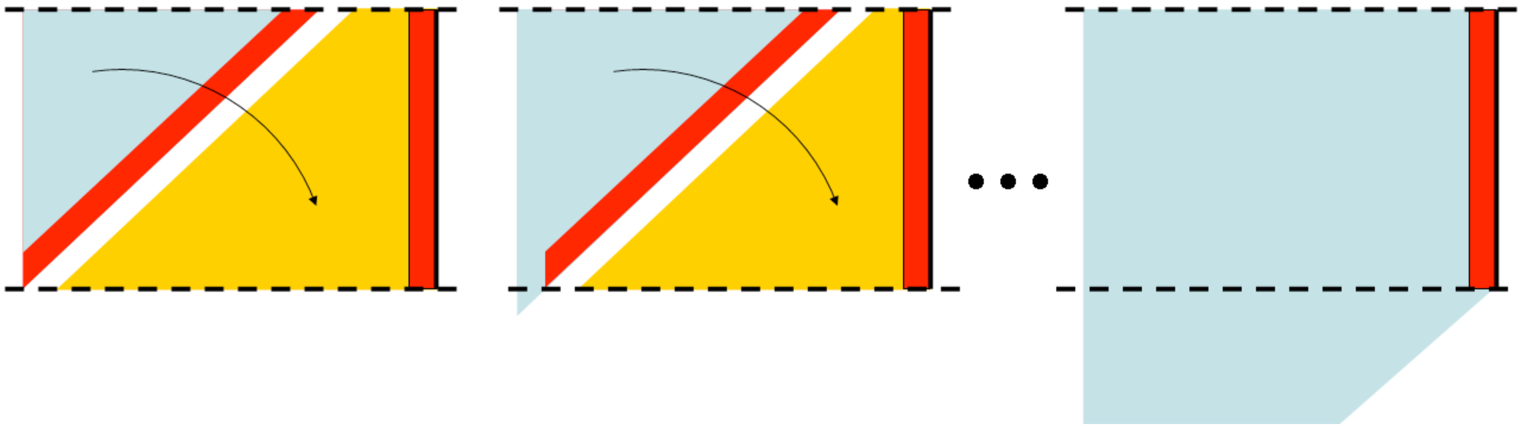
As point out earlier, the (conditional) mean square error of prediction (MSE) is

$$\begin{aligned}\text{mse}_t(\widehat{X}) &= \mathbb{E}\left([X-\widehat{X}]^2|\mathcal{F}_t\right)\\ &= \underbrace{\text{Var}(X|\mathcal{F}_t)}_{\text{process variance}} + \underbrace{\mathbb{E}\left(\mathbb{E}\left(X|\mathcal{F}_t\right)-\widehat{X}\right)^2}_{\text{parameter estimation error}}\end{aligned}$$

i.e $\widehat{X}$ is $\begin{cases}\text{a predictor for } X\\ \text{an estimator for } \mathbb{E}(X|\mathcal{F}_t).\end{cases}$

But this is only a a long-term view, since we focus on the uncertainty *over the whole runoff period*. It is not a one-year solvency view, where we focus on changes *over the next accounting year*.

## Boni-Mali



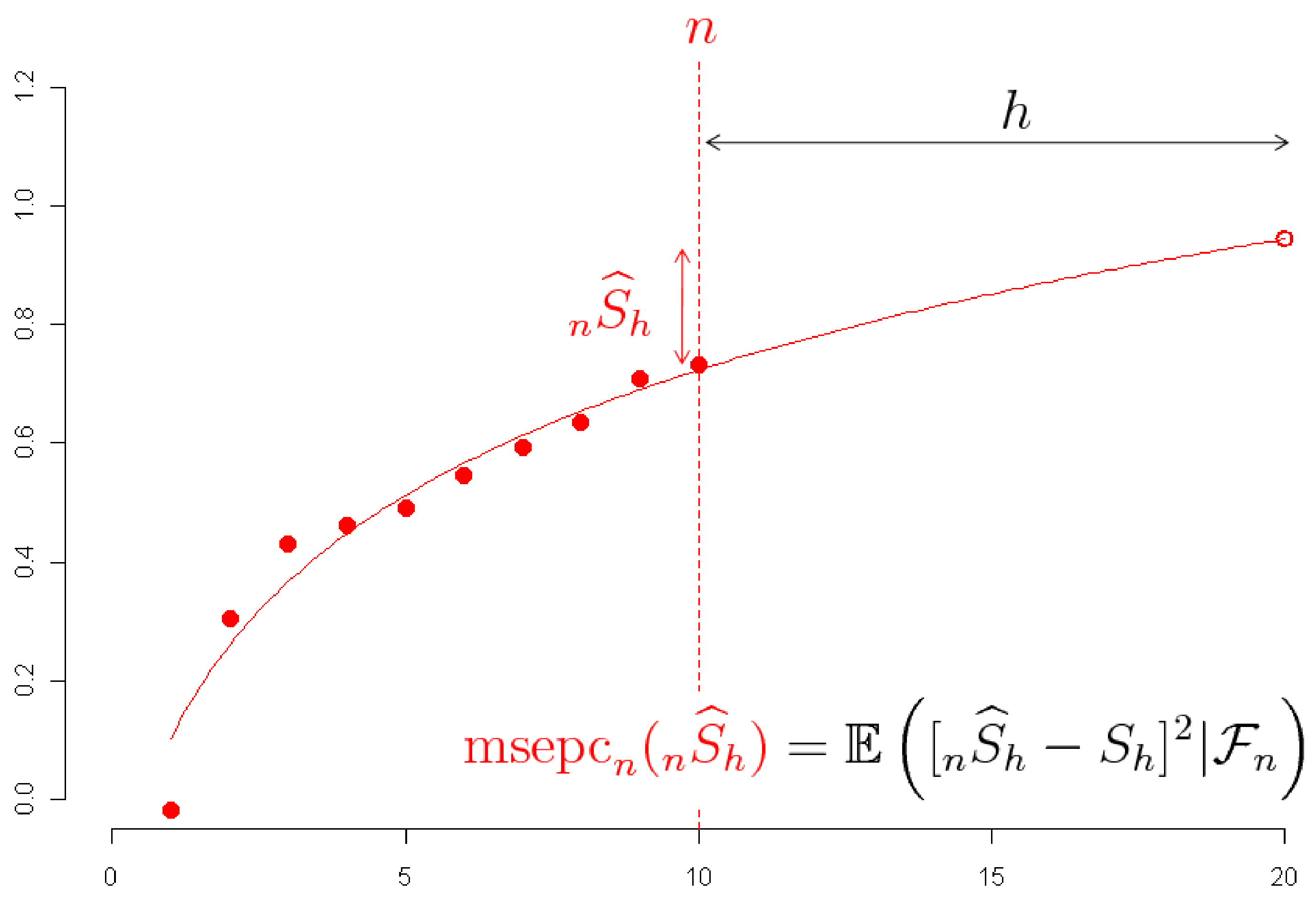
From time $t = n$ and time $t = n + 1$,

$$\widehat{\lambda_j}^{(n)} = \frac{\sum_{i=0}^{n-j-1} C_{i,j+1}}{\sum_{i=0}^{n-j-1} C_{i,j}} \text{ and } \widehat{\lambda_j}^{(n+1)} = \frac{\sum_{i=0}^{n-j} C_{i,j+1}}{\sum_{i=0}^{n-j} C_{i,j}}$$

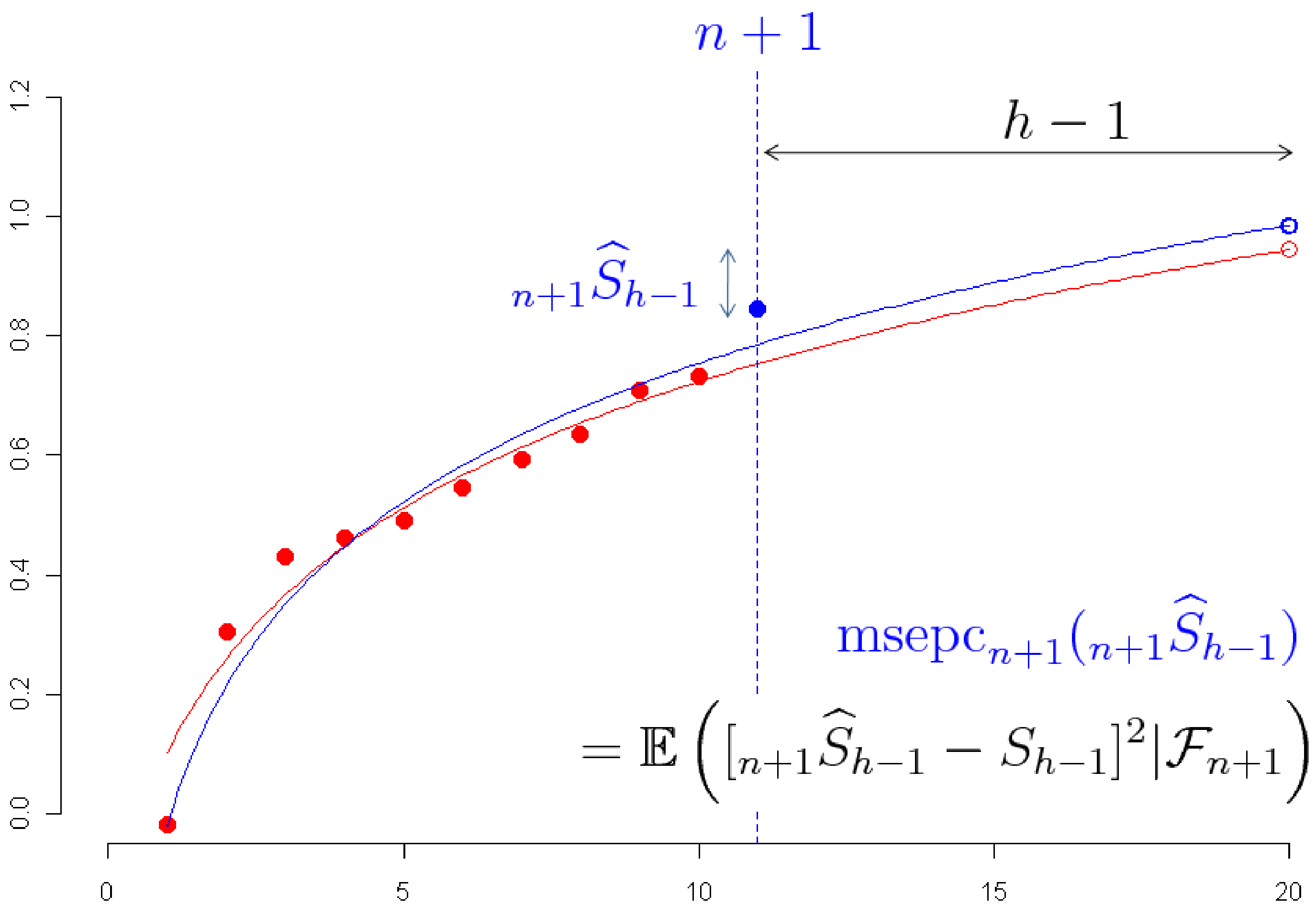
and the ultimate loss predictions are then

$$\widehat{C}_i^{(n)} = C_{i,n-i} \cdot \prod_{j=n-i}^{n} \widehat{\lambda_j}^{(n)} \text{ and } \widehat{C}_i^{(n+1)} = C_{i,n-i+1} \cdot \prod_{j=n-i+1}^{n} \widehat{\lambda_j}^{(n+1)}$$

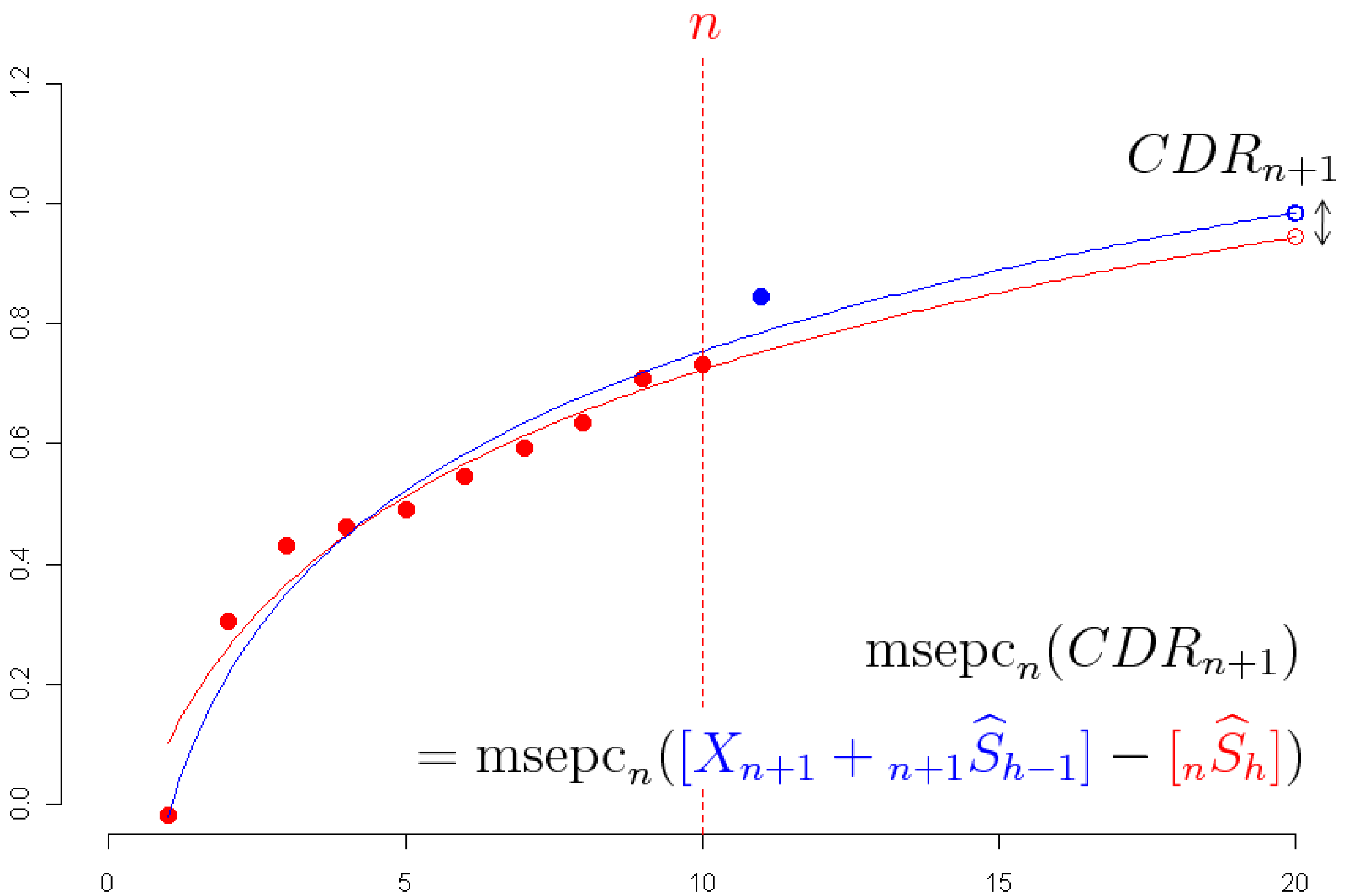
## Boni-Mali and the one-year-uncertainty



## Boni-Mali and the one-year-uncertainty



## Boni-Mali and the one-year-uncertainty



## Boni-Mali

In order to study the one-year claims development, we have to focus on

$$\widehat{R}_i^{(n)} \text{ and } Y_{i,n-i+1} + \widehat{R}_i^{(n+1)}$$

The boni-mali for accident year $i$, from year $n$ to year $n+1$ is then

$$\widehat{BM}_i^{(n,n+1)} = \widehat{R}_i^{(n)} - \left[Y_{i,n-i+1} + \widehat{R}_i^{(n+1)}\right] = \widehat{C}_i^{(n)} - \widehat{C}_i^{(n+1)}.$$

Thus, the conditional one-year runoff uncertainty is

$$\widehat{mse}(\widehat{BM}_i^{(n,n+1)}) = \mathbb{E}\left(\left[\widehat{C}_i^{(n)} - \widehat{C}_i^{(n+1)}\right]^2 |\mathcal{F}_n\right)$$

## Boni-Mali

Hence,

$$\begin{aligned}\widehat{mse}(\widehat{BM}_i^{(n,n+1)}) &= [\widehat{C}_i^{(n)}]^2 \left[ \frac{\widehat{\sigma}^2_{n-i}/[\widehat{\lambda}^{(n)}_{n-i}]^2}{C_{i,n-i}} + \frac{\widehat{\sigma}^2_{n-i}/[\widehat{\lambda}^{(n)}_{n-i}]^2}{\sum_{k=0}^{i-1} C_{k,n-i}} \right. \\ &\left. + \sum_{j=n-i+1}^{n-1} \frac{C_{n-j,j}}{\sum_{k=0}^{n-j} C_{k,j}} \cdot \frac{\widehat{\sigma}^2_j/[\widehat{\lambda}^{(n)}_j]^2}{\sum_{k=0}^{n-j-1} C_{k,j}} \right]\end{aligned}$$

Further, it is possible to derive the MSEP for aggregated accident years (see Merz & Wüthrich (2008)).

## Boni-Mali, Computational Issues

```
> CDR(MackChainLadder(PAID))
            IBNR  CDR(1)S.E.  Mack.S.E.
1        0.00000   0.0000000  0.0000000
2       22.39684   0.6393379  0.6393379
3       35.78388   2.4291919  2.5025153
4       66.06466   4.3969805  5.0459004
5      153.08358  30.9004962 31.3319292
6     2149.65640  60.8243560 68.4489667
Total 2426.98536  72.4127862 79.2954414
```

## Ultimate Loss and Tail Factor

The idea - introduced by Mack (1999) - is to compute

$$\widehat{\lambda}_\infty = \prod_{k\geq n} \widehat{\lambda}_k$$

and then to compute

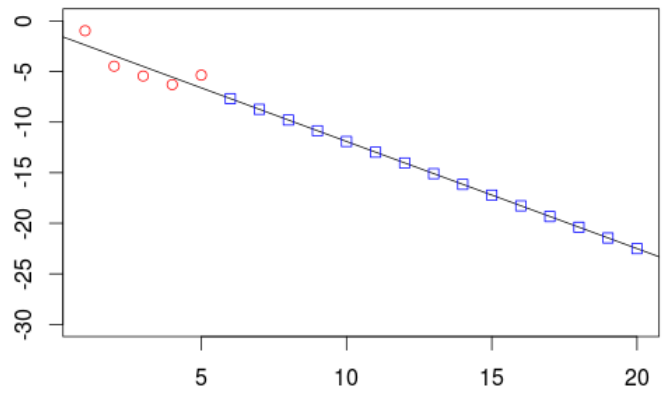
$$C_{i,\infty} = C_{i,n} \times \lambda_\infty.$$

Assume here that $\lambda_i$ are exponentially decaying to 1, i.e. $\log(\lambda_k - 1)$'s are linearly decaying

```
> Lambda=MackChainLadder(PAID)$f[1:(ncol(PAID)-1)]
> logL <- log(Lambda-1)
> tps <- 1:(ncol(PAID)-1)
> modele <- lm(logL~tps)
> logP <- predict(modele,newdata=data.frame(tps=seq(6,1000)))
> (facteur <- prod(exp(logP)+1))
[1] 1.000707
```

## Ultimate Loss and Tail Factor

```
> DIAG <- diag(PAID[,6:1])
> PRODUIT <- c(1,rev(Lambda))
> sum((cumprod(PRODUIT)-1)*DIAG)
[1] 2426.985
> sum((cumprod(PRODUIT)*facteur-1)*DIAG)
[1] 2451.764
```




The ultimate loss is here 0.07% larger, and the reserves are 1% larger.

## Ultimate Loss and Tail Factor

```
> MackChainLadder(Triangle = PAID, tail = TRUE)

   Latest Dev.To.Date Ultimate      IBNR Mack.S.E CV(IBNR)
0   4,456       0.999    4,459      3.15    0.299   0.0948
12  4,730       0.995    4,756     25.76    0.712   0.0277
24  5,420       0.993    5,460     39.64    2.528   0.0638
36  6,020       0.988    6,090     70.37    5.064   0.0720
48  6,794       0.977    6,952    157.99   31.357   0.1985
60  5,217       0.708    7,372  2,154.86   68.499   0.0318

              Totals
Latest:    32,637.00
Dev:            0.93
Ultimate: 35,088.76
IBNR:      2,451.76
Mack.S.E       79.37
CV(IBNR):       0.03
```

## From Chain Ladder to London Chain and London Pivot

La méthode dite London Chain a été introduite par Benjamin & Eagles (1986). On suppose ici que la dynamique des $(C_{ij})_{j=1,..,n}$ est donnée par un modèle de type $AR(1)$ avec constante, de la forme

$$C_{i,k+1} = \lambda_k \cdot C_{ik} + \alpha_k \text{ pour tout } i, k = 1, .., n$$

De façon pratique, on peut noter que la méthode standard de Chain Ladder, reposant sur un modèle de la forme $C_{i,k+1} = \lambda_k C_{ik}$, ne pouvait être appliqué que lorsque les points $(C_{i,k}, C_{i,k+1})$ sont sensiblement alignés (*à $k$ fixé*) sur une droite passant par l'origine. La méthode London Chain suppose elle aussi que les points soient alignés sur une même droite, mais on ne suppose plus qu'elle passe par 0.

**Example**:On obtient la droite passant au mieux par le nuage de points et par 0, et la droite passant au mieux par le nuage de points.

## From Chain Ladder to London Chain and London Pivot

Dans ce modèle, on a alors $2n$ paramètres à identifier : $\lambda_k$ et $\alpha_k$ pour $k = 1, ..., n$. La méthode la plus naturelle consiste à estimer ces paramètres à l'aide des moindres carrés, c'est à dire que l'on cherche, pour tout $k$,

$$\left(\widehat{\lambda}_k, \widehat{\alpha}_k\right) = \text{arg min} \left\{ \sum_{i=1}^{n-k} \left(C_{i,k+1} - \alpha_k - \lambda_k C_{i,k}\right)^2 \right\}$$

ce qui donne, finallement

$$\widehat{\lambda}_k = \frac{\frac{1}{n-k}\sum_{i=1}^{n-k} C_{i,k}C_{i,k+1} - \overline{C}_k^{(k)}\overline{C}_{k+1}^{(k)}}{\frac{1}{n-k}\sum_{i=1}^{n-k} C_{i,k}^2 - \overline{C}_k^{(k)2}}$$

$$\text{où } \overline{C}_k^{(k)} = \frac{1}{n-k}\sum_{i=1}^{n-k} C_{i,k} \text{ et } \overline{C}_{k+1}^{(k)} = \frac{1}{n-k}\sum_{i=1}^{n-k} C_{i,k+1}$$

et où la constante est donnée par $\widehat{\alpha}_k = \overline{C}_{k+1}^{(k)} - \widehat{\lambda}_k \overline{C}_k^{(k)}$.

## From Chain Ladder to London Chain and London Pivot

Dans le cas particulier du triangle que nous étudions, on obtient

| $k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\widehat{\lambda}_k$ | 1.404 | 1.405 | 1.0036 | 1.0103 | 1.0047 |
| $\widehat{\alpha}_k$ | $-90.311$ | $-147.27$ | 3.742 | $-38.493$ | 0 |

## From Chain Ladder to London Chain and London Pivot

The completed (cumulated) triangle is then

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | |
| 2 | 3871 | 5345 | 5398 | 5420 | | |
| 3 | 4239 | 5917 | 6020 | | | |
| 4 | 4929 | 6794 | | | | |
| 5 | 5217 | | | | | |

## From Chain Ladder to London Chain and London Pivot

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 3209 | 4372 | 4411 | 4428 | 4435 | 4456 |
| 1 | 3367 | 4659 | 4696 | 4720 | 4730 | 4752 |
| 2 | 3871 | 5345 | 5398 | 5420 | 5437 | 5463 |
| 3 | 4239 | 5917 | 6020 | 6045 | 6069 | 6098 |
| 4 | 4929 | 6794 | 6922 | 6950 | 6983 | 7016 |
| 5 | 5217 | 7234 | 7380 | 7410 | 7447 | 7483 |

One the triangle has been completed, we obtain the amount of reserves, with respectively 22, 43, 78, 222 and 2266 per accident year, i.e. the total is 2631 (to be compared with 2427, obtained with the Chain Ladder technique).

## From Chain Ladder to London Chain and London Pivot

La méthode dite London Pivot a été introduite par Straub, dans *Nonlife Insurance Mathematics* (1989). On suppose ici que $C_{i,k+1}$ et $C_{i,k}$ sont liés par une relation de la forme

$$C_{i,k+1} + \alpha = \lambda_k.(C_{i,k} + \alpha)$$

(de façon pratique, les points $(C_{i,k}, C_{i,k+1})$ doivent être sensiblement alignés (*à* $k$ *fixé*) sur une droite passant par le point dit pivot $(-\alpha, -\alpha)$). Dans ce modèle, $(n+1)$ paramètres sont alors a estimer, et une estimation par moindres carrés ordinaires donne des estimateurs de façon itérative.

## Introduction to factorial models: Taylor (1977)

This approach was studied in a paper entitled *Separation of inflation and other effects from the distribution of non-life insurance claim delays*

We assume the incremental payments are functions of two factors, one related to accident year $i$, and one to calendar year $i+j$. Hence, assume that

$$Y_{ij} = r_j \mu_{i+j-1} \text{ for all } i, j$$

$$\begin{array}{ccccc}
r_1\mu_1 & r_2\mu_2 & \cdots & r_{n-1}\mu_{n-1} & r_n\mu_n \\
r_1\mu_2 & r_2\mu_3 & \cdots & r_{n-1}\mu_n & \\
\vdots & \vdots & & & \\
r_1\mu_{n-1} & r_2\mu_n & & & \\
r_1\mu_n & & & &
\end{array}
\qquad
\begin{array}{ccccc}
r_1\mu_1 & r_2\mu_2 & \cdots & r_{n-1}\mu_{n-1} & r_n\mu_n \\
r_1\mu_2 & r_2\mu_3 & \cdots & r_{n-1}\mu_n & \\
\vdots & \vdots & & & \\
r_1\mu_{n-1} & r_2\mu_n & & & \\
r_1\mu_n & & & &
\end{array}$$

Hence, incremental payments are functions of development factors, $r_j$, and a calendar factor, $\mu_{i+j-1}$, that might be related to some inflation index.

## Introduction to factorial models: Taylor (1977)

In order to identify factors $r_1, r_2, .., r_n$ and $\mu_1, \mu_2, ..., \mu_n$, i.e. $2n$ coefficients, an additional constraint is necessary, e.g. on the $r_j$'s, $r_1 + r_2 + .... + r_n = 1$ (this will be called arithmetic separation method). Hence, the sum on the latest diagonal is

$$d_n = Y_{1,n} + Y_{2,n-1} + ... + Y_{n,1} = \mu_n \left(r_1 + r_2 + .... + r_k\right) = \mu_n$$

On the first sur-diagonal

$$d_{n-1} = Y_{1,n-1} + Y_{2,n-2} + ... + Y_{n-1,1} = \mu_{n-1} \left(r_1 + r_2 + .... + r_{n-1}\right) = \mu_{n-1} \left(1 - r_n\right)$$

and using the $n$th column, we get $\gamma_n = Y_{1,n} = r_n \mu_n$, so that

$$r_n = \frac{\gamma_n}{\mu_n} \text{ and } \mu_{n-1} = \frac{d_{n-1}}{1 - r_n}$$

More generally, it is possible to iterate this idea, and on the $i$th sur-diagonal,

$$\begin{aligned} d_{n-i} &= Y_{1,n-i} + Y_{2,n-i-1} + ... + Y_{n-i,1} = \mu_{n-i} \left(r_1 + r_2 + .... + r_{n-i}\right) \\ &= \mu_{n-i} \left(1 - [r_n + r_{n-1} + ... + r_{n-i+1}]\right) \end{aligned}$$

## Introduction to factorial models: Taylor (1977)

and finally, based on the $n-i+1$th column,

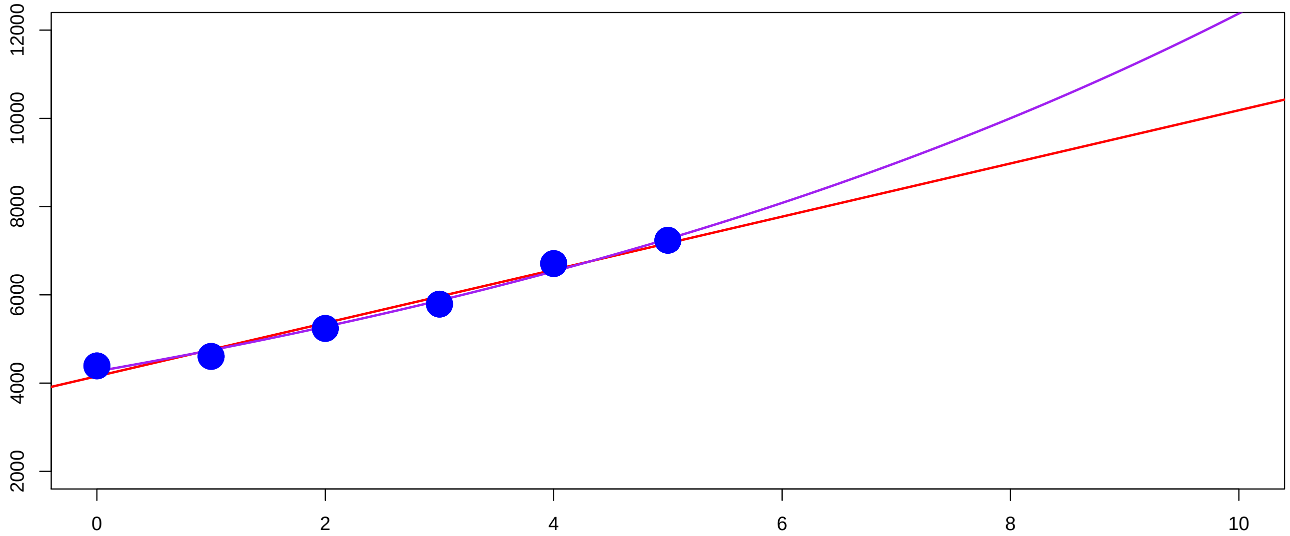
$$\begin{aligned}\gamma_{n-i+1} &= Y_{1,n-i+1}+Y_{2,n-i+1}+\ldots+Y_{i-1,n-i+1}\\ &= r_{n-i+1}\mu_{n-i+1}+\ldots+r_{n-i+1}\mu_{n-1}+r_{n-i+1}\mu_{n}\end{aligned}$$

$$r_{n-i+1}=\frac{\gamma_{n-i+1}}{\mu_n+\mu_{n-1}+\ldots+\mu_{n-i+1}} \text{ and } \mu_{k-i}=\frac{d_{n-i}}{1-[r_n+r_{n-1}+\ldots+r_{n-i+1}]}$$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\mu_k$ | 4391 | 4606 | 5240 | 5791 | 6710 | 7238 |
| $r_k$ in % | 73.08 | 25.25 | 0.93 | 0.32 | 0.12 | 0.29 |

## Introduction to factorial models: Taylor (1977)

The challenge here is to forecast forecast values for the $\mu_k$'s. Either a linear model or an exponential model can be considered.



## Lemaire (1982) and autoregressive models

Instead of a *simple* Markov process, it is possible to assume that the $C_{i,j}$'s can be modeled with an autorgressive model in two directions, rows and columns,

$$C_{i,j} = \alpha C_{i-1,j} + \beta C_{i,j-1} + \gamma.$$

## Zehnwirth (1977)

Here, we consider the following model for the $C_{i,j}$'s

$$C_{i,j} = \exp\left(\alpha_i + \gamma_i \cdot j\right)\left(1+j\right)^{\beta_i},$$

which can be seen as an extended Gamma model. $\alpha_i$ is a scale parameter, while $\beta_i$ and $\gamma_i$ are shape parameters. Note that

$$\log C_{i,j} = \alpha_i + \beta_i \log\left(1+j\right) + \gamma_i \cdot j.$$

For convenience, we usually assume that $\beta_i = \beta$ et $\gamma_i = \gamma$.

Note that if $\log C_{i,j}$ is assume to be Gaussian, then $C_{i,j}$ will be lognormal. But then, estimators one the $C_{i,j}$'s while be overestimated.

## Zehnwirth (1977)

Assume that $\log C_{i,j} \sim \mathcal{N}\left(X_{i,j}\beta, \sigma^2\right)$, then, if parameters were obtained using maximum likelihood techniques

$$\begin{aligned}
\mathbb{E}\left(\widehat{C}_{i,j}\right) &= \mathbb{E}\left(\exp\left(X_{i,j}\widehat{\beta} + \frac{\widehat{\sigma}^2}{2}\right)\right) \\
&= C_{i,j}\exp\left(-\frac{n-1}{n}\frac{\sigma^2}{2}\right)\left(1 - \frac{\sigma^2}{n}\right)^{-\frac{n-1}{2}} > C_{i,j},
\end{aligned}$$

Further, the homoscedastic assumption might not be relevant. Thus Zehnwirth suggested

$$\sigma^2_{i,j} = Var\left(\log C_{i,j}\right) = \sigma^2\left(1+j\right)^h.$$

## Regression and reserving

De Vylder (1978) proposed a least squares factor method, i.e. we need to find $\boldsymbol{\alpha} = (\alpha_0, \cdots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_0, \cdots, \beta_n)$ such

$$(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}) = \text{argmin} \sum_{i,j=0}^{n} \left(Y_{i,j} - \alpha_i \times \beta_j\right)^2,$$

or equivalently, assume that

$$Y_{i,j} \sim \mathcal{N}(\alpha_i \times \beta_j, \sigma^2), \text{ independent.}$$

A more general model proposed by De Vylder is the following

$$(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\gamma}}) = \text{argmin} \sum_{i,j=0}^{n} \left(Y_{i,j} - \alpha_i \times \beta_j \times \gamma_{i+j-1}\right)^2.$$

In order to have an identifiable model, De Vylder suggested to assume $\gamma_k = \gamma^k$ (so that this coefficient will be related to some inflation index).