Probit transformation for nonparametric kernel estimation of the copula

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http://arxiv.org/abs/1404.4414

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Motivation

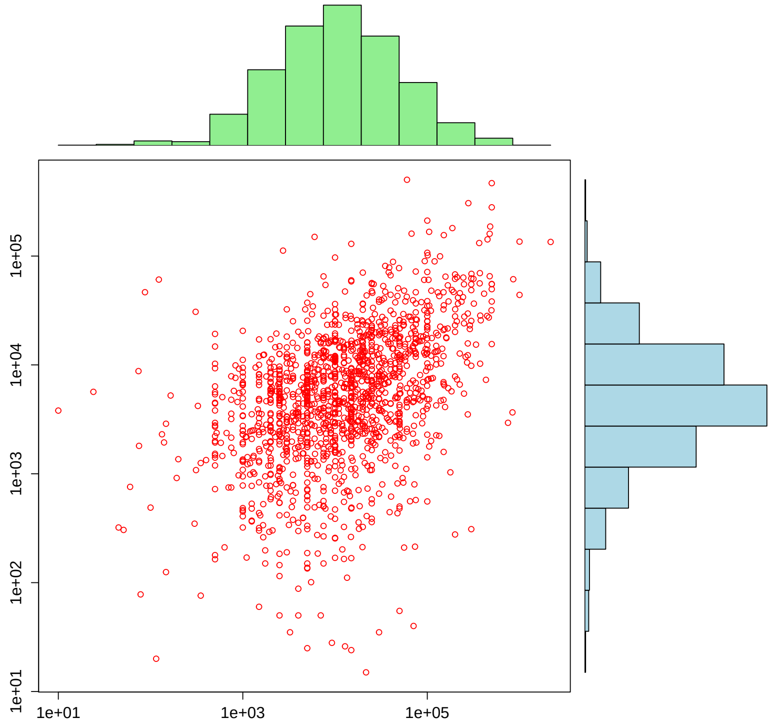
Consider some n-i.i.d. sample $\{(X_i, Y_i)\}$ with cumulative distribution function F_{XY} and joint density f_{XY}. Let F_X and F_Y denote the marginal distributions, and C the copula,

$$F_{XY}(x,y) = C(F_X(x), F_Y(y))$$

so that

$$f_{XY}(x,y) = f_X(x) f_Y(y) c(F_X(x), F_Y(y))$$

We want a nonparametric estimate of c on $[0,1]^2$.



Notations

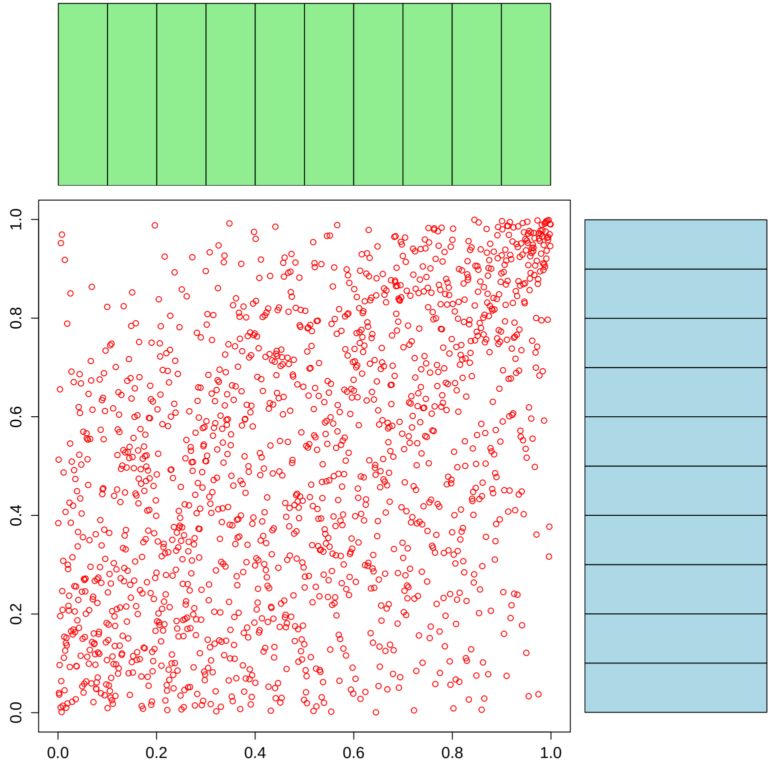
Define uniformized n-i.i.d. sample $\{(U_i, V_i)\}$

$$U_i = F_X(X_i) \text{ and } V_i = F_Y(Y_i)$$

or uniformized n-i.i.d. pseudo-sample $\{(\hat{U}_i, \hat{V}_i)\}$

$$\hat{U}_i = \frac{n}{n+1}\hat{F}_{Xn}(X_i) \text{ and } \hat{V}_i = \frac{n}{n+1}\hat{F}_{Yn}(Y_i)$$

where $\hat{F}_{Xn}$ and $\hat{F}_{Yn}$ denote empirical c.d.f.



Standard Kernel Estimate

The standard kernel estimator for c, say $\hat{c}^*$, at $(u,v)\in\mathcal{I}$ would be (see Wand & Jones (1995))

$$\hat{c}^*(u,v)=\frac{1}{n|\mathbf{H}_{UV}|^{1/2}}\sum_{i=1}^{n}\mathbf{K}\left(\mathbf{H}_{UV}^{-1/2}\begin{pmatrix}u-U_i\\ v-V_i\end{pmatrix}\right),\qquad (1)$$

where $\mathbf{K}:\mathbb{R}^2\to\mathbb{R}$ is a kernel function and $\mathbf{H}_{UV}$ is a bandwidth matrix.

Standard Kernel Estimate

However, this estimator is not consistent along boundaries of $[0,1]^2$

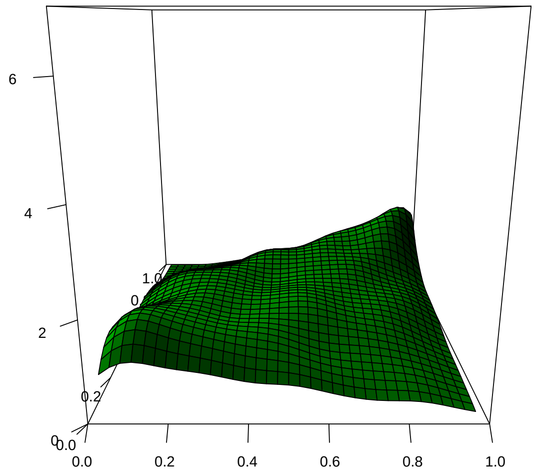
$$\mathbb{E}(\hat{c}^*(u,v)) = \frac{1}{4}c(u,v) + O(h) \text{ at corners}$$

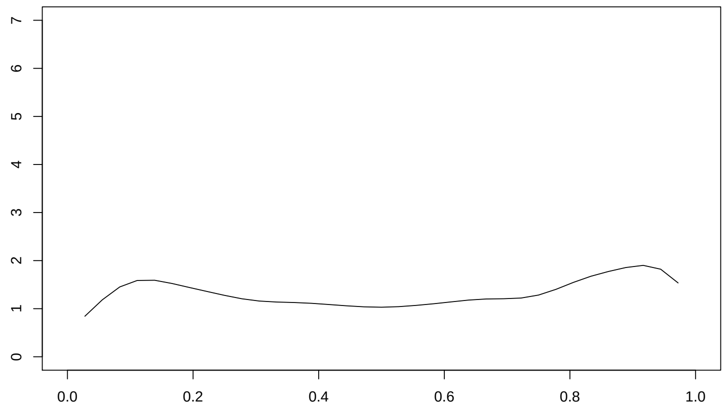
$$\mathbb{E}(\hat{c}^*(u,v)) = \frac{1}{2}c(u,v) + O(h) \text{ on the borders}$$

if $\mathbf{K}$ is symmetric and $\mathbf{H}_{UV}$ symmetric.

Corrections have been proposed, e.g. mirror reflection Gijbels (1990) or the usage of boundary kernels Chen (2007), but with mixed results.

Remark: the graph on the bottom is $\hat{c}^*$ on the (first) diagonal.



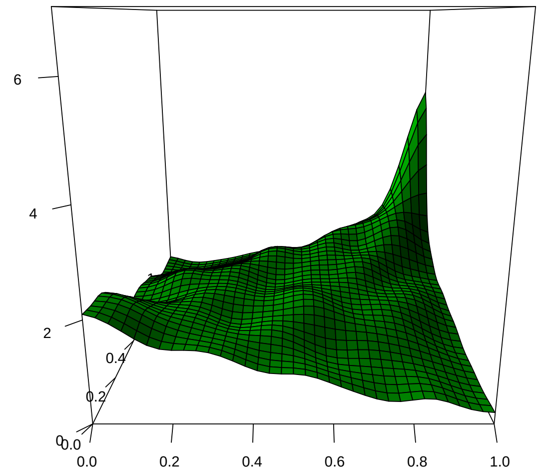


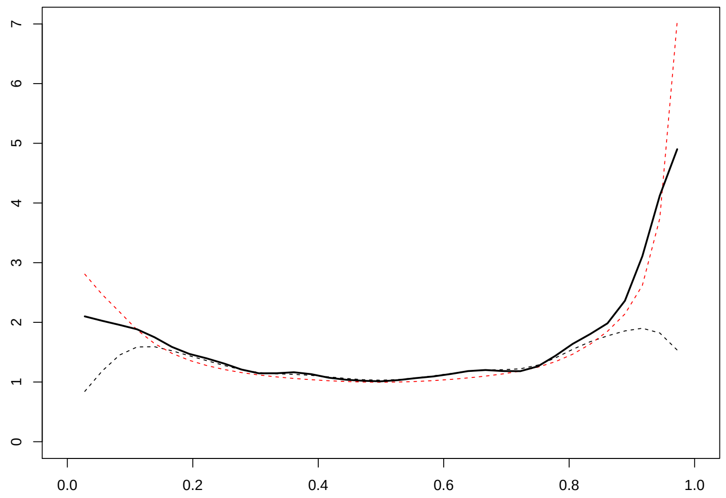
Mirror Kernel Estimate

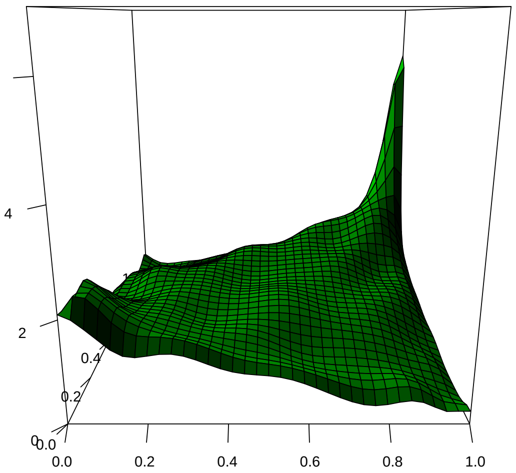
Use an enlarged sample: instead of only $\{(\hat{U}_i, \hat{V}_i)\}$, add $\{(-\hat{U}_i, \hat{V}_i)\}$, $\{(\hat{U}_i, -\hat{V}_i)\}$, $\{(-\hat{U}_i, -\hat{V}_i)\}$, $\{(\hat{U}_i, 2-\hat{V}_i)\}$, $\{(2-\hat{U}_i, \hat{V}_i)\}$,$\{(-\hat{U}_i, 2-\hat{V}_i)\}$, $\{(2-\hat{U}_i, -\hat{V}_i)\}$ and $\{(2-\hat{U}_i, 2-\hat{V}_i)\}$.

See Gijbels & Mielniczuk (1990).

That estimator will be used as a benchmark in the simulation study.





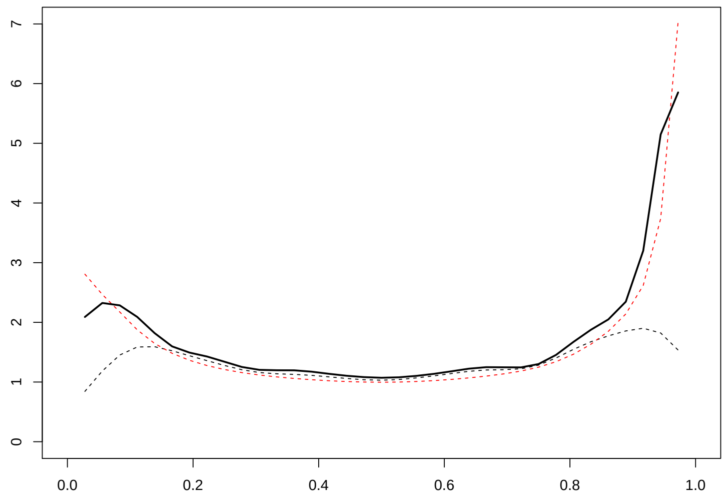


Using Beta Kernels

Use a Kernel which is a product of beta kernels

$$\mathbf{K}_{\boldsymbol{x}_i}(\boldsymbol{u})) \propto \left(u_1^{\frac{x_{1,i}}{b}} [1-u_1]^{\frac{x_{1,i}}{b}} \right) \cdot \left(u_2^{\frac{x_{2,i}}{b}} [1-u_2]^{\frac{x_{2,i}}{b}} \right)$$

See Chen (1999).



Probit Transformation

See Devroye & Győfi (1985) and Marron & Ruppert (1994).

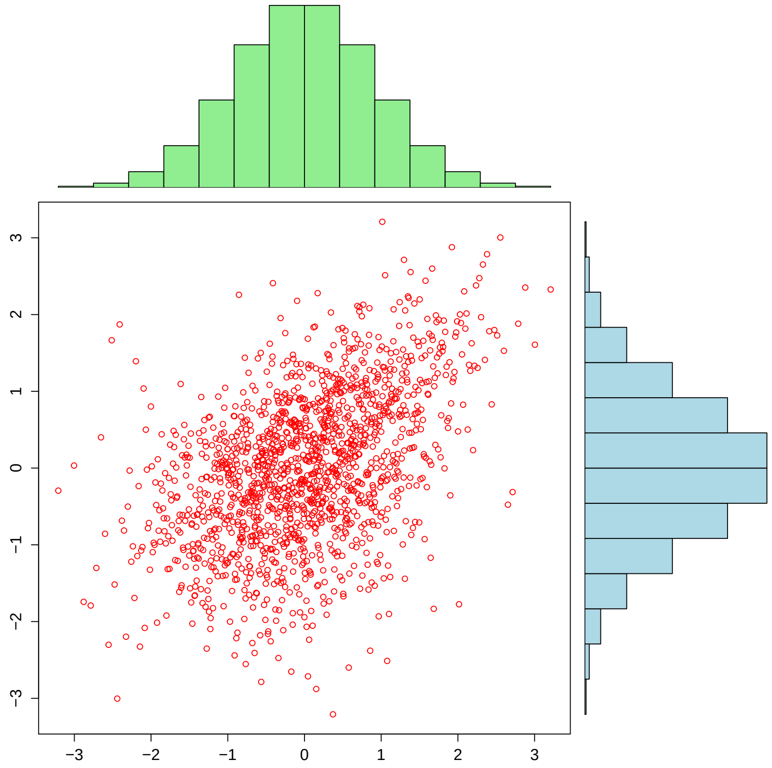
Define normalized n-i.i.d. sample $\{(S_i, T_i)\}$

$$S_i = \Phi^{-1}(U_i) \text{ and } T_i = \Phi^{-1}(V_i)$$

or normalized n-i.i.d. pseudo-sample $\{(\hat{S}_i, \hat{T}_i)\}$

$$\hat{U}_i = \Phi^{-1}(\hat{U}_i) \text{ and } \hat{V}_i = \Phi^{-1}(\hat{V}_i)$$

where Φ^{-1} is the quantile function of $\mathcal{N}(0,1)$ (probit transformation).



Probit Transformation

$$F_{ST}(x,y)=C(\Phi(x),\Phi(y))$$

so that

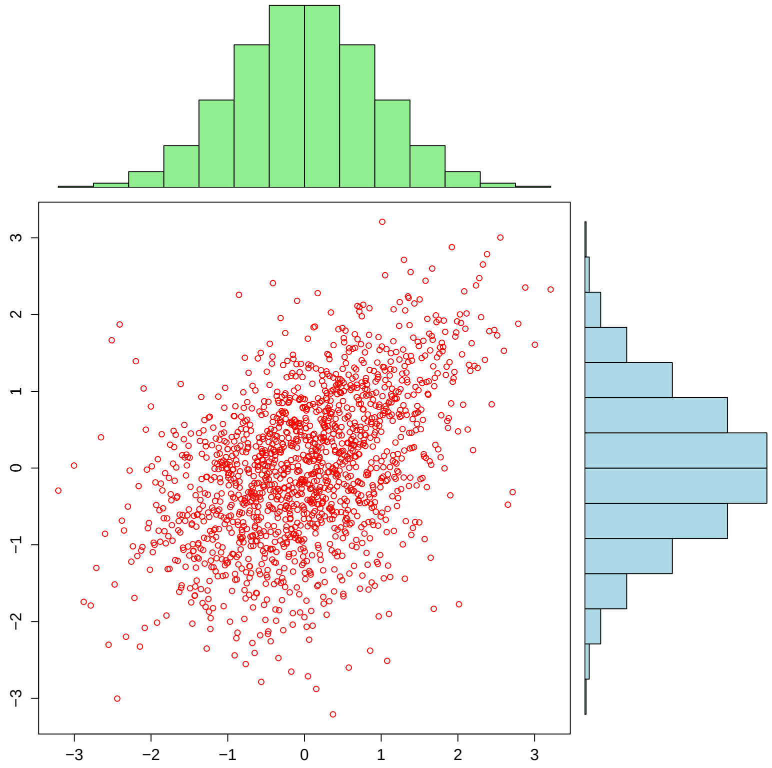
$$f_{ST}(x,y)=\phi(x)\phi(y)c(\Phi(x),\Phi(y))$$

Thus

$$c(u,v)=\frac{f_{ST}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}.$$

So use

$$\hat{c}^{(\tau)}(u,v)=\frac{\hat{f}_{ST}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$



The naive estimator

Since we cannot use

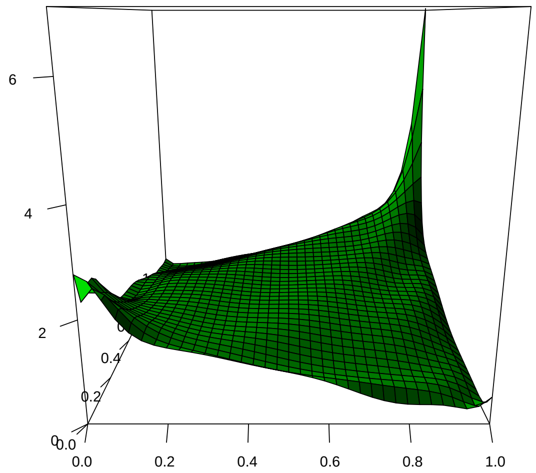
$$\hat{f}^*_{ST}(s,t) = \frac{1}{n|\mathbf{H}_{ST}|^{1/2}} \sum_{i=1}^{n} \mathbf{K}\left(\mathbf{H}_{ST}^{-1/2}\begin{pmatrix} s - S_i \\ t - T_i \end{pmatrix}\right),$$

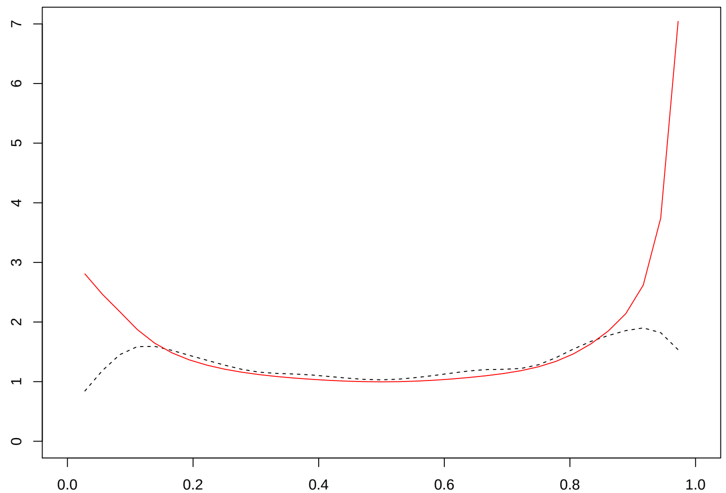
where $\mathbf{K}$ is a kernel function and $\mathbf{H}_{ST}$ is a bandwidth matrix, use

$$\hat{f}_{ST}(s,t) = \frac{1}{n|\mathbf{H}_{ST}|^{1/2}} \sum_{i=1}^{n} \mathbf{K}\left(\mathbf{H}_{ST}^{-1/2}\begin{pmatrix} s - \hat{S}_i \\ t - \hat{T}_i \end{pmatrix}\right).$$

and the copula density is

$$\hat{c}^{(\tau)}(u,v) = \frac{\hat{f}_{ST}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$





The naive estimator

$$\hat{c}^{(\tau)}(u,v) = \frac{1}{n|\mathbf{H}_{ST}|^{1/2}\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}\sum_{i=1}^{n}\mathbf{K}\left(\mathbf{H}_{ST}^{-1/2}\begin{pmatrix}\Phi^{-1}(u)-\Phi^{-1}(\hat{U}_i)\\ \Phi^{-1}(v)-\Phi^{-1}(\hat{V}_i)\end{pmatrix}\right)$$

as suggested in C., Fermanian & Scaillet (2007) and Lopez-Paz . *et al.* (2013).

Note that Omelka . *et al.* (2009) obtained theoretical properties on the convergence of $\hat{C}^{(\tau)}(u,v)$ (not c).

Improved probit-transformation copula density estimators

When estimating a density from **pseudo-sample**, Loader (1996) and Hjort & Jones (1996) define a local likelihood estimator

Around $(s,t) \in \mathbb{R}^2$, use a polynomial approximation of order p for $\log f_{ST}$

$$\log f_{ST}(\check{s},\check{t}) \simeq a_{1,0}(s,t) + a_{1,1}(s,t)(\check{s}-s) + a_{1,2}(s,t)(\check{t}-t) \doteq P_{\mathbf{a}_1}(\check{s}-s,\check{t}-t)$$

$$\begin{aligned}\log f_{ST}(\check{s},\check{t}) \simeq\ & a_{2,0}(s,t) + a_{2,1}(s,t)(\check{s}-s) + a_{2,2}(s,t)(\check{t}-t) \\ & + a_{2,3}(s,t)(\check{s}-s)^2 + a_{2,4}(s,t)(\check{t}-t)^2 + a_{2,5}(s,t)(\check{s}-s)(\check{t}-t) \\ & \doteq P_{\mathbf{a}_2}(\check{s}-s,\check{t}-t).\end{aligned}$$

Improved probit-transformation copula density estimators

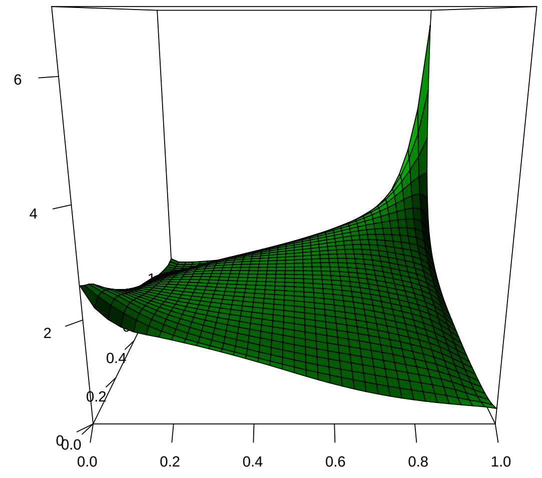
Remark Vectors $\mathbf{a}_1(s,t) = (a_{1,0}(s,t), a_{1,1}(s,t), a_{1,2}(s,t))$ and $\mathbf{a}_2(s,t) \doteq (a_{2,0}(s,t), \ldots, a_{2,5}(s,t))$ are then estimated by solving a weighted maximum likelihood problem.

$$\tilde{\mathbf{a}}_p(s,t) = \arg\max_{\mathbf{a}_p} \left\{ \sum_{i=1}^{n} \mathbf{K}\left(\mathbf{H}_{ST}^{-1/2} \begin{pmatrix} s - \hat{S}_i \\ t - \hat{T}_i \end{pmatrix} \right) P_{\mathbf{a}_p}(\hat{S}_i - s, \hat{T}_i - t) \right.$$
$$\left. - n \iint_{\mathbb{R}^2} \mathbf{K}\left(\mathbf{H}_{ST}^{-1/2} \begin{pmatrix} s - \check{s} \\ t - \check{t} \end{pmatrix} \right) \exp\left(P_{\mathbf{a}_p}(\check{s} - s, \check{t} - t) \right) d\check{s}\, d\check{t} \right\},$$

The estimate of f_{ST} at (s,t) is then $\tilde{f}_{ST}^{(p)}(s,t) = \exp(\tilde{a}_{p,0}(s,t))$, for $p = 1, 2$.

The Improved probit-transformation kernel copula density estimators are

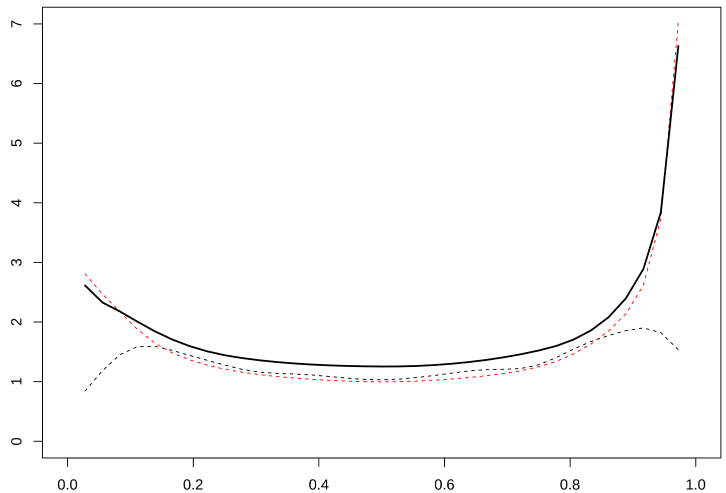
$$\tilde{c}^{(\tau,p)}(u,v) = \frac{\tilde{f}_{ST}^{(p)}(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$



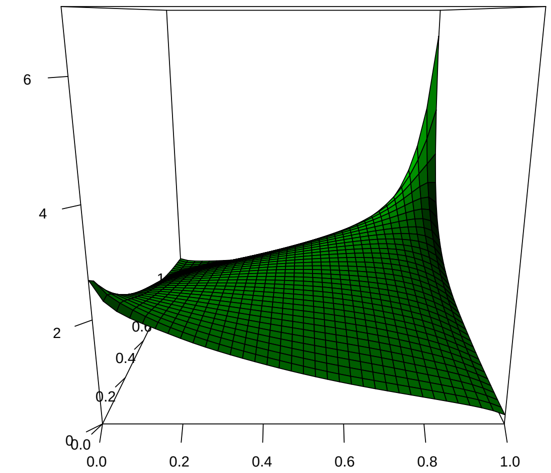
Improved probit-transformation copula density estimators

For the local log-linear ($p = 1$) approximation

$$\tilde{c}^{(\tau,1)}(u,v) = \frac{\exp(\tilde{a}_{1,0}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$

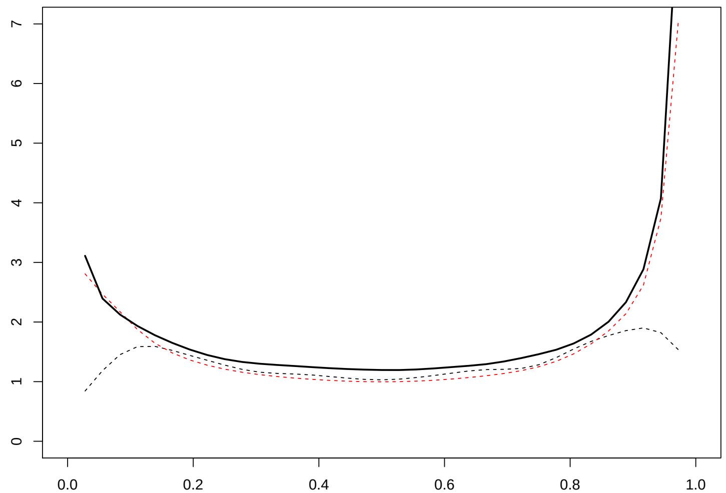


Improved probit-transformation copula density estimators



For the local log-quadratic ($p = 2$) approximation

$$\tilde{c}^{(\tau,2)}(u,v) = \frac{\exp(\tilde{a}_{2,0}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$



Asymptotic properties

A1. The sample $\{(X_i, Y_i)\}$ is a n- i.i.d. sample from the joint distribution F_{XY}, an absolutely continuous distribution with marginals F_X and F_Y strictly increasing on their support;

(uniqueness of the copula)

Asymptotic properties

A2. The copula C of F_{XY} is such that $(\partial C/\partial u)(u,v)$ and $(\partial^2 C/\partial u^2)(u,v)$ exist and are continuous on $\{(u,v) : u \in (0,1), v \in [0,1]\}$, and $(\partial C/\partial v)(u,v)$ and $(\partial^2 C/\partial v^2)(u,v)$ exist and are continuous on $\{(u,v) : u \in [0,1], v \in (0,1)\}$. In addition, there are constants K_1 and K_2 such that

$$\begin{cases} \left|\dfrac{\partial^2 C}{\partial u^2}(u,v)\right| \leq \dfrac{K_1}{u(1-u)} & \text{for } (u,v) \in (0,1) \times [0,1]; \\ \left|\dfrac{\partial^2 C}{\partial v^2}(u,v)\right| \leq \dfrac{K_2}{v(1-v)} & \text{for } (u,v) \in [0,1] \times (0,1); \end{cases}$$

A3. The density c of C exists, is positive and admits continuous second-order partial derivatives on the interior of the unit square $\mathcal{I}$. In addition, there is a constant K_{00} such that

$$c(u,v) \leq K_{00} \min\left(\frac{1}{u(1-u)}, \frac{1}{v(1-v)}\right) \qquad \forall (u,v) \in (0,1)^2.$$

see Segers (2012).

Asymptotic properties

Assume that $\mathbf{K}(z_1, z_2) = \phi(z_1)\phi(z_2)$ and $\mathbf{H}_{ST} = h^2\mathbf{I}$ with $h \sim n^{-a}$ for some $a \in (0, 1/4)$. Under Assumptions A1-A3, the 'naive' probit transformation kernel copula density estimator at any $(u, v) \in (0, 1)^2$ is such that

$$\sqrt{nh^2}\left(\hat{c}^{(\tau)}(u,v) - c(u,v) - h^2\frac{b(u,v)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma^2(u,v)\right),$$

$$\begin{aligned}\text{where } b(u,v) = \frac{1}{2}\bigg\{&\frac{\partial^2 c}{\partial u^2}(u,v)\phi^2(\Phi^{-1}(u)) + \frac{\partial^2 c}{\partial v^2}(u,v)\phi^2(\Phi^{-1}(v)) \\ &- 3\left(\frac{\partial c}{\partial u}(u,v)\Phi^{-1}(u)\phi(\Phi^{-1}(u)) + \frac{\partial c}{\partial v}(u,v)\Phi^{-1}(v)\phi(\Phi^{-1}(v))\right) \\ &+ c(u,v)\left(\{\Phi^{-1}(u)\}^2 + \{\Phi^{-1}(v)\}^2 - 2\right)\bigg\} \qquad (2)\end{aligned}$$

and $\sigma^2(u,v) = \dfrac{c(u,v)}{4\pi\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$.

The Amended version

The last unbounded term in b be easily adjusted.

$$\hat{c}^{(\tau\text{am})}(u,v)=\frac{\hat{f}_{ST}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}\times\frac{1}{1+\frac{1}{2}h^2\left(\{\Phi^{-1}(u)\}^2+\{\Phi^{-1}(v)\}^2-2\right)}.$$

The asymptotic bias becomes proportional to

$$b^{(\text{am})}(u,v)=\frac{1}{2}\left\{\frac{\partial^2 c}{\partial u^2}(u,v)\phi^2(\Phi^{-1}(u))+\frac{\partial^2 c}{\partial v^2}(u,v)\phi^2(\Phi^{-1}(v))\right.$$
$$\left.-3\left(\frac{\partial c}{\partial u}(u,v)\Phi^{-1}(u)\phi(\Phi^{-1}(u))+\frac{\partial c}{\partial v}(u,v)\Phi^{-1}(v)\phi(\Phi^{-1}(v))\right)\right\}.$$

A local log-linear probit-transformation kernel estimator

$$\tilde{c}^{*(\tau,1)}(u,v) = \tilde{f}^{*(1)}_{ST}(\Phi^{-1}(u),\Phi^{-1}(v)) / \left(\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))\right)$$

Then

$$\sqrt{nh^2}\left(\tilde{c}^{*(\tau,1)}(u,v) - c(u,v) - h^2\frac{b^{(1)}(u,v)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0,\sigma^{(1)\,2}(u,v)\right),$$

$$\begin{aligned}\text{where}\quad b^{(1)}(u,v) = \frac{1}{2}\Bigg\{&\frac{\partial^2 c}{\partial u^2}(u,v)\phi^2(\Phi^{-1}(u)) + \frac{\partial^2 c}{\partial v^2}(u,v)\phi^2(\Phi^{-1}(v))\\ &- \frac{1}{c(u,v)}\left(\left\{\frac{\partial c}{\partial u}(u,v)\right\}^2\phi^2(\Phi^{-1}(u)) + \left\{\frac{\partial c}{\partial v}(u,v)\right\}^2\phi^2(\Phi^{-1}(v))\right)\\ &- \left(\frac{\partial c}{\partial u}(u,v)\Phi^{-1}(u)\phi(\Phi^{-1}(u)) + \frac{\partial c}{\partial v}(u,v)\Phi^{-1}(v)\phi(\Phi^{-1}(v))\right) - 2c(u,v)\Bigg\}\end{aligned}$$

Using a higher order polynomial approximation

Locally fitting a polynomial of a higher degree is known to reduce the asymptotic bias of the estimator, here from order $O(h^2)$ to order $O(h^4)$, see Loader (1996) or Hjort (1996), under sufficient smoothness conditions.

If f_{ST} admits continuous fourth-order partial derivatives and is positive at (s,t), then

$$\sqrt{nh^2}\left(\tilde{f}^{*(2)}_{ST}(s,t) - f_{ST}(s,t) - h^4 b^{(2)}_{ST}(s,t)\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, {\sigma^{(2)}_{ST}}^2(s,t)\right),$$

where ${\sigma^{(2)}_{ST}}^2(s,t) = \dfrac{5}{2}\,\dfrac{f_{ST}(s,t)}{4\pi}$ and

$$b^{(2)}_{ST}(s,t) = -\frac{1}{8} f_{ST}(s,t)$$

$$\times\left\{\left(\frac{\partial^4 g}{\partial s^4} + \frac{\partial^4 g}{\partial t^4}\right) + 4\left(\frac{\partial^3 g}{\partial s^3}\frac{\partial g}{\partial s} + \frac{\partial^3 g}{\partial t^3}\frac{\partial g}{\partial t} + \frac{\partial^3 g}{\partial s^2\partial t}\frac{\partial g}{\partial t} + \frac{\partial^3 g}{\partial s\partial t^2}\frac{\partial g}{\partial s}\right) + 2\frac{\partial^4 g}{\partial s^2 \partial t^2}\right\}(s,t),$$

with $g(s,t) = \log f_{ST}(s,t)$.

Using a higher order polynomial approximation

A4. The copula density $c(u,v) = (\partial^2 C/\partial u \partial v)(u,v)$ admits continuous fourth-order partial derivatives on the interior of the unit square $[0,1]^2$.

Then

$$\sqrt{nh^2}\left(\tilde{c}^{*(\tau,2)}(u,v) - c(u,v) - h^4 \frac{b^{(2)}(u,v)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, {\sigma^{(2)}}^2(u,v)\right)$$

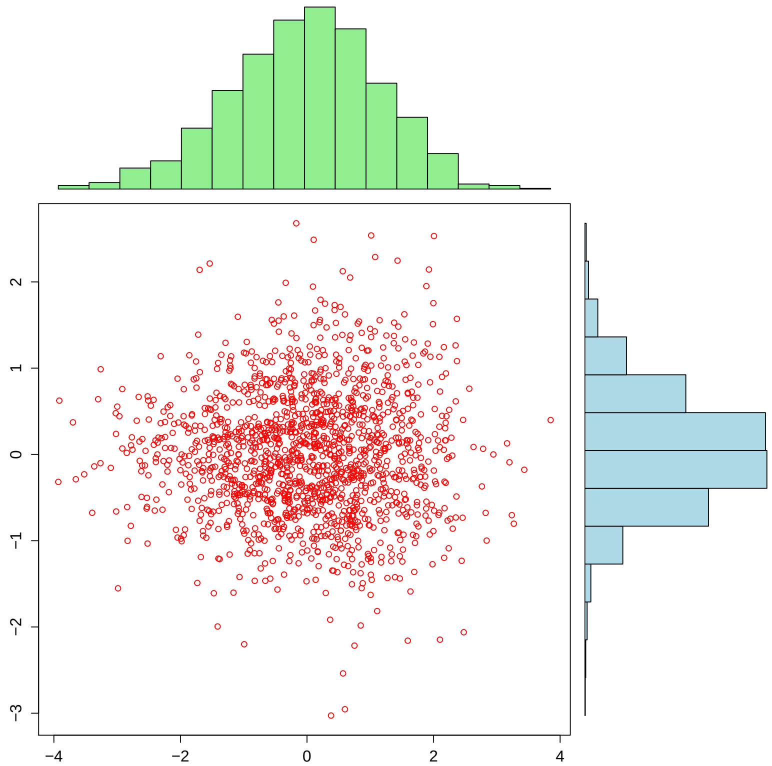
where ${\sigma^{(2)}}^2(u,v) = \dfrac{5}{2}\dfrac{c(u,v)}{4\pi\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$

Improving Bandwidth choice

Consider the principal components decomposition of the $(n \times 2)$ matrix $[\hat{\boldsymbol{S}}, \hat{\boldsymbol{T}}] = \boldsymbol{M}$.

Let $W_1 = (W_{11}, W_{12})^\top$ and $W_2 = (W_{21}, W_{22})^\top$ be the eigenvectors of $\boldsymbol{M}^\top \boldsymbol{M}$. Set

$$\begin{pmatrix} Q \\ R \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} S \\ T \end{pmatrix} = \mathbf{W} \begin{pmatrix} S \\ T \end{pmatrix}$$



which is only a linear reparametrization of $\mathbb{R}^2$, so an estimate of f_{ST} can be readily obtained from an estimate of the density of (Q, R)

Since $\{\hat{Q}_i\}$ and $\{\hat{R}_i\}$ are empirically uncorrelated, consider a diagonal bandwidth matrix $\mathbf{H}_{QR} = \text{diag}(h_Q^2, h_R^2)$.

Improving Bandwidth choice

Use univariate procedures to select h_Q and h_R independently

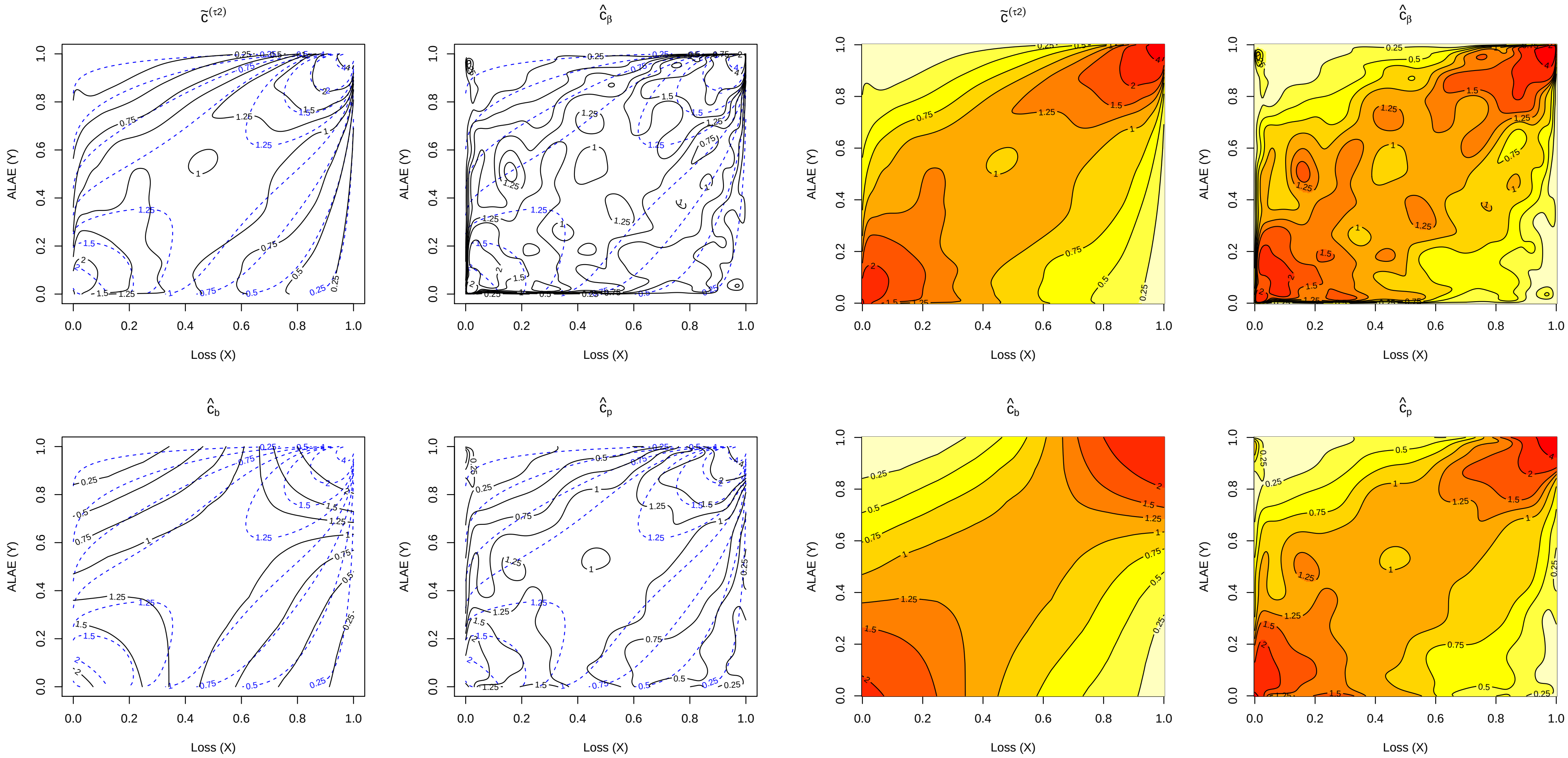
Denote $\tilde{f}_Q^{(p)}$ and $\tilde{f}_R^{(p)}$ $(p = 1, 2)$, the local log-polynomial estimators for the densities

h_Q can be selected via cross-validation (see Section 5.3.3 in Loader (1999))

$$h_Q = \arg\min_{h>0} \left\{ \int_{-\infty}^{\infty} \left\{ \tilde{f}_Q^{(p)}(q) \right\}^2 dq - \frac{2}{n} \sum_{i=1}^{n} \tilde{f}_{Q(-i)}^{(p)}(\hat{Q}_i) \right\},$$

where $\tilde{f}_{Q(-i)}^{(p)}$ is the 'leave-one-out' version of $\tilde{f}_Q^{(p)}$.

Graphical Comparison (loss ALAE dataset)



Simulation Study

$M = 1,000$ independent random samples $\{(U_i, V_i)\}_{i=1}^n$ of sizes $n = 200$, $n = 500$ and $n = 1000$ were generated from each of the following copulas:

- the independence copula (i.e., U_i's and V_i's drawn independently);
- the Gaussian copula, with parameters $\rho = 0.31$, $\rho = 0.59$ and $\rho = 0.81$;
- the Student t-copula with 4 degrees of freedom, with parameters $\rho = 0.31$, $\rho = 0.59$ and $\rho = 0.81$;
- the Frank copula, with parameter $\theta = 1.86$, $\theta = 4.16$ and $\theta = 7.93$;
- the Gumbel copula, with parameter $\theta = 1.25$, $\theta = 1.67$ and $\theta = 2.5$;
- the Clayton copula, with parameter $\theta = 0.5$, $\theta = 1.67$ and $\theta = 2.5$.

(approximated) MISE relative to the MISE of the mirror-reflection estimator (last column), $n = 1000$. Bold values show the minimum MISE for the corresponding copula (non-significantly different values are highlighted as well).

$n = 1000$	$\hat{c}^{(\tau)}$	$\hat{c}^{(\tau\text{am})}$	$\tilde{c}^{(\tau,1)}$	$\tilde{c}^{(\tau,2)}$	$\hat{c}_1^{(\beta)}$	$\hat{c}_2^{(\beta)}$	$\hat{c}_1^{(B)}$	$\hat{c}_2^{(B)}$	$\hat{c}_1^{(p)}$	$\hat{c}_2^{(p)}$	$\hat{c}_3^{(p)}$
Indep	3.57	2.80	2.89	1.40	7.96	11.65	1.69	3.43	1.62	0.50	**0.14**
Gauss2	2.03	1.52	1.60	0.76	4.63	6.06	1.10	1.82	0.98	**0.66**	0.89
Gauss4	0.63	0.49	0.44	**0.21**	1.72	1.60	0.75	0.58	0.62	0.99	2.93
Gauss6	0.21	0.20	0.11	**0.05**	0.74	0.33	0.77	0.37	0.72	1.21	2.83
Std(4)2	0.61	0.56	0.50	**0.40**	1.57	1.80	0.78	0.67	0.75	1.01	1.88
Std(4)4	0.21	0.27	**0.17**	**0.15**	0.88	0.51	0.75	0.42	0.75	1.12	2.07
Std(4)6	**0.09**	0.17	**0.08**	**0.09**	0.70	0.19	0.82	0.47	0.90	1.17	1.90
Frank2	3.31	2.42	2.57	1.35	7.16	9.63	1.70	2.95	1.31	**0.45**	**0.49**
Frank4	2.35	1.45	1.51	0.99	4.42	4.89	1.49	1.65	**0.60**	0.72	6.14
Frank6	0.96	0.52	**0.45**	**0.44**	1.51	1.19	1.35	0.76	0.65	1.58	7.25
Gumbel2	0.65	0.62	0.56	**0.43**	1.77	1.97	0.82	0.75	0.83	1.03	1.52
Gumbel4	**0.18**	0.28	**0.16**	**0.19**	0.89	0.41	0.78	0.47	0.81	1.10	1.78
Gumbel6	**0.09**	0.21	**0.10**	0.15	0.78	0.29	0.85	0.58	0.94	1.12	1.63
Clayton2	0.63	0.60	0.51	**0.34**	1.78	1.99	0.78	0.70	0.79	1.04	1.79
Clayton4	**0.11**	0.26	**0.10**	**0.15**	0.79	0.27	0.83	0.56	0.90	1.10	1.50
Clayton6	**0.11**	0.28	**0.08**	0.15	0.82	0.35	0.88	0.67	0.96	1.09	1.36