Optional reinsurance with ruin probability target

Arthur Charpentier

http://blogperso.univ-rennes1.fr/arthur.charpentier/
Ruin, solvency and reinsurance

“reinsurance plays an important role in reducing the risk in an insurance portfolio.”


“reinsurance is able to offer additional underwriting capacity for cedants, but also to reduce the probability of a direct insurer’s ruin.”

Proportional Reinsurance (Quota-Share)

- claim loss $X : \alpha X$ paid by the cedant, $(1 - \alpha)X$ paid by the reinsurer,
- premium $P : \alpha P$ kept by the cedant, $(1 - \alpha)P$ transferred to the reinsurer,

Nonproportional Reinsurance (Excess-of-Loss)

- claim loss $X : \min\{X, u\}$ paid by the cedant, $\max\{0, X - u\}$ paid by the reinsurer,
- premium $P : P_u$ kept by the cedant, $P - P_u$ transferred to the reinsurer,
Propotional versus nonproportional reinsurance

**Fig. 1** – Reinsurance mechanism for claims indemnity, proportional versus non-proportional treaties.
Mathematical framework

Classical Cramér-Lundberg framework:

- claims arrival is driven by an homogeneous Poisson process, $N_t \sim \mathcal{P}(\lambda t)$,
  durations between consecutive arrivals $T_{i+1} - T_i$ are independent $\mathcal{E}(\lambda)$,
- claims size $X_1, \cdots, X_n, \cdots$ are i.i.d. non-negative random variables, independent of claims arrival.

Let $Y_t = \sum_{i=1}^{N_t} X_i$ denote the aggregate amount of claims during period $[0, t]$. 
**Premium**

The pure premium required over period \([0, t]\) is

\[
\pi_t = \mathbb{E}(Y_t) = \mathbb{E}(N_t)\mathbb{E}(X) = \lambda \mathbb{E}(X) t.
\]

Note that more general premiums can be considered, e.g.

- safety loading proportional to the pure premium, \(\pi_t = [1 + \lambda] \cdot \mathbb{E}(Y_t)\),
- safety loading proportional to the variance, \(\pi_t = \mathbb{E}(Y_t) + \lambda \cdot Var(Y_t)\),
- safety loading proportional to the standard deviation, \(\pi_t = \mathbb{E}(Y_t) + \lambda \cdot \sqrt{Var(Y_t)}\),
- entropic premium (exponential expected utility) \(\pi_t = \frac{1}{\alpha} \log \left(\mathbb{E}(e^{\alpha Y_t})\right)\),
- Esscher premium \(\pi_t = \frac{\mathbb{E}(X \cdot e^{\alpha Y_t})}{\mathbb{E}(e^{\alpha Y_t})}\),
- Wang distorted premium \(\pi_t = \int_0^\infty \Phi \left(\Phi^{-1} \left(\mathbb{P}(Y_t > x)\right) + \lambda\right) dx\),
A classical solvency problem

Given a ruin probability target, e.g. 0.1%, on a given time horizon $T$, find capital $u$ such that,

$$
\psi(T, u) = 1 - \mathbb{P}(u + \pi t \geq Y_t, \forall t \in [0, T]) \\
= 1 - \mathbb{P}(S_t \geq 0 \forall t \in [0, T]) \\
= \mathbb{P}(\inf\{S_t\} < 0) = 0.1%,$$

where $S_t = u + \pi t - Y_t$ denotes the insurance company surplus.
A classical solvency problem

After reinsurance, the net surplus is then

\[ S_t^{(\theta)} = u + \pi^{(\theta)} t - \sum_{i=1}^{N_t} X_i^{(\theta)}, \]

where \( \pi^{(\theta)} = \mathbb{E} \left( \sum_{i=1}^{N_1} X_i^{(\theta)} \right) \) and

\[
\begin{cases} 
X_i^{(\theta)} = \theta X_i, & \theta \in [0, 1], \text{ for quota share treaties}, \\
X_i^{(\theta)} = \min\{\theta, X_i\}, & \theta > 0, \text{ for excess-of-loss treaties}.
\end{cases}
\]
Classical answers : using upper bounds

Instead of targeting a ruin probability level, Centeno (1986) and Chapter 9 in Dickson (2005) target an upper bound of the ruin probability.

In the case of light tailed claims, let $\gamma$ denote the “adjustment coefficient”, defined as the unique positive root of

$$\lambda + \pi \gamma = \lambda M_X(\gamma), \text{ where } M_X(t) = \mathbb{E}(\exp(tX)).$$

The Lundberg inequality states that

$$0 \leq \psi(T, u) \leq \psi(\infty, u) \leq \exp[-\gamma u] = \psi_{CL}(u).$$

Gerber (1976) proposed an improvement in the case of finite horizon ($T < \infty$).
Classical answers : using approximations $u \to \infty$

de Vylder (1996) proposed the following approximation, assuming that $\mathbb{E}(|X|^3) < \infty$,

$$
\psi_{dV}(u) \sim \frac{1}{1 + \theta'} \exp \left( -\frac{\beta' \theta' \mu}{1 + \theta'} \right) \text{ quand } u \to \infty
$$

where

$$
\theta' = \frac{2\mu m_3}{3m_2^2} \theta \text{ et } \beta' = \frac{3m_2}{m_3}.
$$

Beekman (1969) considered

$$
\psi_B(u) \frac{1}{1 + \theta} [1 - \Gamma(u)] \text{ quand } u \to \infty
$$

where $\Gamma$ is the c.d.f. of the $\Gamma(\alpha, \beta)$ distribution

$$
\alpha = \frac{1}{1 + \theta} \left( 1 + \left( \frac{4\mu m_3}{3m_2^2} - 1 \right) \theta \right) \text{ et } \beta = 2\mu \theta \left( m_2 + \left( \frac{4\mu m_3}{3m_2^2} - m_2 \right) \theta \right)^{-1}
$$
Classical answers : using approximations \( u \to \infty \)

Rényi - see Grandell (2000) - proposed an exponential approximation of the convoluted distribution function

\[
\psi_R(u) \sim \frac{1}{1 + \theta} \exp \left( - \frac{2\mu \theta u}{m_2 (1 + \theta)} \right) \text{ quand } u \to \infty
\]

In the case of subexponential claims

\[
\psi_{SE}(u) \sim \frac{1}{\theta \mu} \left( \mu - \int_0^u F(x) \, dx \right)
\]
### Classical answers: using approximations $u \to \infty$

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Proportional reinsurance (QS)

With proportional reinsurance, if $1 - \alpha$ is the ceding ratio,

$$S_t^{(\alpha)} = u + \alpha \pi t - \sum_{i=1}^{N_t} \alpha X_i = (1 - \alpha)u + \alpha S_t$$

Reinsurance can always decrease ruin probability.

Assuming that there was ruin (without reinsurance) before time $T$, if the insurance had ceded a proportion $1 - \alpha^*$ of its business, where

$$\alpha^* = \frac{u}{u - \inf\{S_t, t \in [0, T]\}},$$

there would have been no ruin (at least on the period $[0, T]$).

$$\alpha^* = \frac{u}{u - \min\{S_t, t \in [0, T]\}} \mathbf{1}(\min\{S_t, t \in [0, T]\} < 0) + \mathbf{1}(\min\{S_t, t \in [0, T]\} \geq 0),$$

then

$$\psi(T, u, \alpha) = \psi(T, u) \cdot \mathbb{P}(\alpha^* \leq \alpha).$$
Fig. 2 – Proportional reinsurance used to decrease ruin probability, the plain line is the brut surplus, and the dotted line the cedant surplus with a reinsurance treaty.
Proportional reinsurance (QS)

In that case, the algorithm to plot the ruin probability as a function of the reinsurance share is simply the following

```r
RUIN <- 0; ALPHA <- NA
for(i in 1:Nb.Simul){
  T <- rexp(N,lambda); T <- T[ cumsum(T)<1 ]; n <- length(T)
  X <- r.claims(n); S <- u+premium*cumsum(T)-cumsum(X)
  if(min(S)<0) { RUIN <- RUIN +1
    ALPHA <- c(ALPHA,u/(u-min(S))) }
}
rate <- seq(0,1,by=.01); proportion <- rep(NA,length(rate))
for(i in 1:length(rate)){
  proportion[i]=sum(ALPHA<rate[i])/length(ALPHA)
}
plot(rate,proportion*RUIN/Nb.Simul)
```
Proportional reinsurance (QS)

Fig. 3 – Ruin probability as a function of the cedant’s share.
Proportional reinsurance (QS)

Fig. 4 – Ruin probability as a function of the cedant’s share.
Nonproportional reinsurance (QS)

With nonproportional reinsurance, if $d \geq 0$ is the priority of the reinsurance contract, the surplus process for the company is

$$S_t^{(d)} = u + \pi^{(d)} t - \sum_{i=1}^{N_t} \min\{X_i, d\} \text{ where } \pi^{(d)} = \mathbb{E}(S_1^{(d)}) = \mathbb{E}(N_1) \cdot \mathbb{E}(\min\{X_i, d\}) .$$

Here the problem is that it is possible to have a lot of small claims (smaller than $d$), and to have ruin with the reinsurance cover (since $p^{(d)} < p$ and $\min\{X_i, d\} = X_i$ for all $i$ if claims are no very large), while there was no ruin without the reinsurance cover (see Figure 5).
Proportional reinsurance (QS)

Impact of nonproportional reinsurance in case of nonruin

Fig. 5 – Case where nonproportional reinsurance can cause ruin, the plain line is the brut surplus, and the dotted line the cedant surplus with a reinsurance treaty.
Fig. 6 – Monte Carlo computation of ruin probabilities, where $n = 10,000$ trajectories are generated for each deductible, with a 95% confidence interval.
Références


