Solvency II' *newspeak*
'one year uncertainty for IBNR'
the boostrap approach

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Lyon Summer School, July 2010

Agenda of the talk

- **Solvency II : CP 71 and the *one year horizon***
- **Solvency II : new way of looking at the 'uncertainty'**
  - From MSE to MSEP (MSE of prediction)
  - From MSEP to MSEPC (conditional MSEP)
  - CDR, claims development result
- **From Mack (1993) to Merz & Wüthrich (2009)**
- **Updating Poisson-ODP bootstrap technique**

	one year	ultimate
China ladder	Merz & Wüthrich (2008)	Mack (1993)
GLM+boostrap	×	Hacheleister & Stanard (1975)
		England & Verrall (1999)

‘one year horizon for the reserve risk’

AISAM-ACME study on non-life long tail liabilities

Reserve risk and risk margin assessment under Solvency II

17 October 2007

'one year horizon for the reserve risk'

4 The concept of the one year horizon for the reserve risk

The uncertainty measurement of reserves in the balance sheet (called risk margin in the Solvency II framework) and the reserve risk do not have the same time horizon. It seems important to underline this point because it may be a source of confusion when the calibration is discussed.

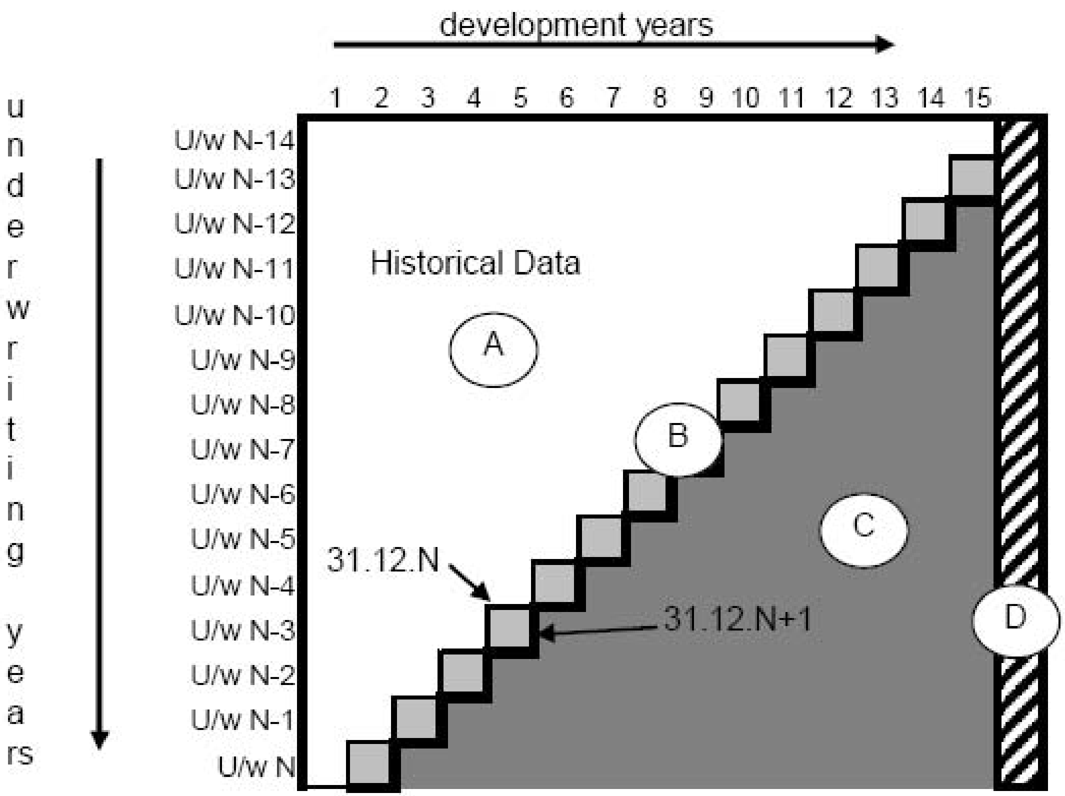
4.1.2 The reserve risk captures uncertainty over a one year period

4.1.2.1 The Solvency II draft Directive framework

The SCR has the following definition[3]:

> *"The SCR corresponds to the economic capital a (re)insurance undertaking needs to hold in order to limit the probability of ruin to 0.5%, i.e. ruin would occur once every 200 years (see Article 100). The SCR is calculated using Value-at-Risk techniques, either in accordance with the standard formula, or using an internal model: all potential losses, including adverse revaluation of assets and liabilities over the next 12 months are to be assessed. The SCR reflects the true risk profile of the undertaking, taking account of all quantifiable risks, as well as the net impact of risk mitigation techniques."*

'one year horizon for the reserve risk'



‘one year horizon for the reserve risk’

	Process error (intrisic volatilitity)			Estimation error (model error)			Prediction error (total)		
	Whole run-off	One year horizon	Variation (%)	Whole run-off	One year horizon	Variation (%)	Whole run-off	One year horizon	Variation (%)
participant n°1 (WCp1)	4.60%	4.34%	*-6%*	2.10%	1.81%	*-14%*	5.10%	4.70%	*-8%*
participant n°1 (WCp2)	1.48%	1.23%	*-17%*	1.45%	1.30%	*-10%*	2.07%	1.79%	*-14%*
participant n°2 (GL1)	4.40%	1.90%	*-57%*	6.60%	3.00%	*-55%*	7.90%	3.60%	*-54%*
participant n°2 (GL2)	4.80%	2.50%	*-48%*	6.80%	3.20%	*-53%*	8.30%	4.10%	*-51%*
participant n°3 (GL)	4.65%	2.54%	*-45%*	6.15%	2.80%	*-54%*	7.70%	3.78%	*-51%*
participant n°5 (GL)	5.23%	2.03%	*-61%*	9.19%	4.96%	*-46%*	10.58%	5.36%	*-49%*
participant n°5 (WCp)	6.91%	5.56%	*-20%*	5.51%	3.42%	*-38%*	8.84%	6.53%	*-26%*
participant n°9 (GL)	6.80%	4.80%	*-29%*	11.60%	6.60%	*-43%*	13.50%	8.20%	*-39%*
participant n°10 (GL)	5.05%	3.77%	*-25%*	3.62%	3.17%	*-12%*	6.21%	4.93%	*-21%*

'one year horizon for the reserve risk'

CEIOPS
Committee of European Insurance and Occupational Pensions Supervisors

Consultation Paper No. 71

CEIOPS-CP-71-09
2 November 2009

Draft CEIOPS' Advice for
Level 2 Implementing Measures on
Solvency II:

SCR Standard Formula

Calibration of non-life underwriting risk

‘one year horizon for the reserve risk’

Method 4

3.242 This approach is consistent with the undertaking specific estimate assumptions from the Technical Specifications for QIS4 for premium risk.

3.243 This method involves a three stage process:

a. **Involves by undertaking calculating the mean squared error of prediction of the claims development result over the one year.**

- The mean squared errors are calculated using the approach detailed in “Modelling The Claims Development Result For Solvency Purposes” by Michael Merz and Mario V Wuthrich, Casualty Actuarial Society E-Forum, Fall 2008.
- Furthermore, in the claims triangles:
- cumulative payments $C_{i,j}$ in different accident years i are independent
- for each accident year, the cumulative payments $(C_{i,j})_j$ are a Markov process and there are constants f_j and s_j such that $E(C_{i,j}|C_{i,j-1})=f_jC_{i,j-1}$ and $Var(C_{i,j}|C_{i,j-1})=s_j^2C_{i,j-1}$.

Standard models in IBNR models

- Chain Ladder $C_{i,j+1} = \lambda_j \cdot C_{i,j}$
-
-

Standard models in IBNR models

- Chain Ladder $C_{i,j+1} = \lambda_j \cdot C_{i,j}$
- Factor models $Y_{i,j} = \varphi(A_i, B_j)$
-

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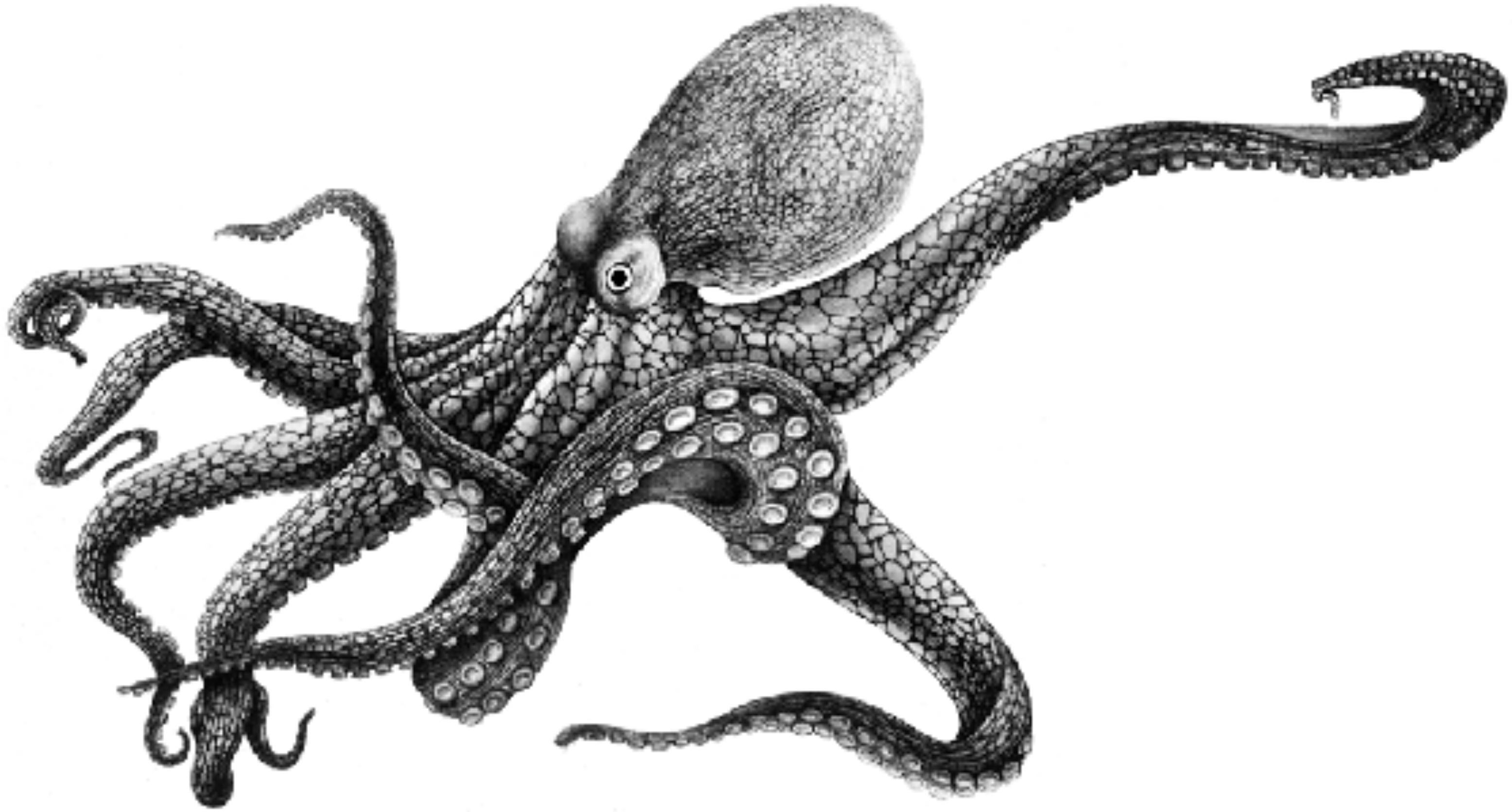

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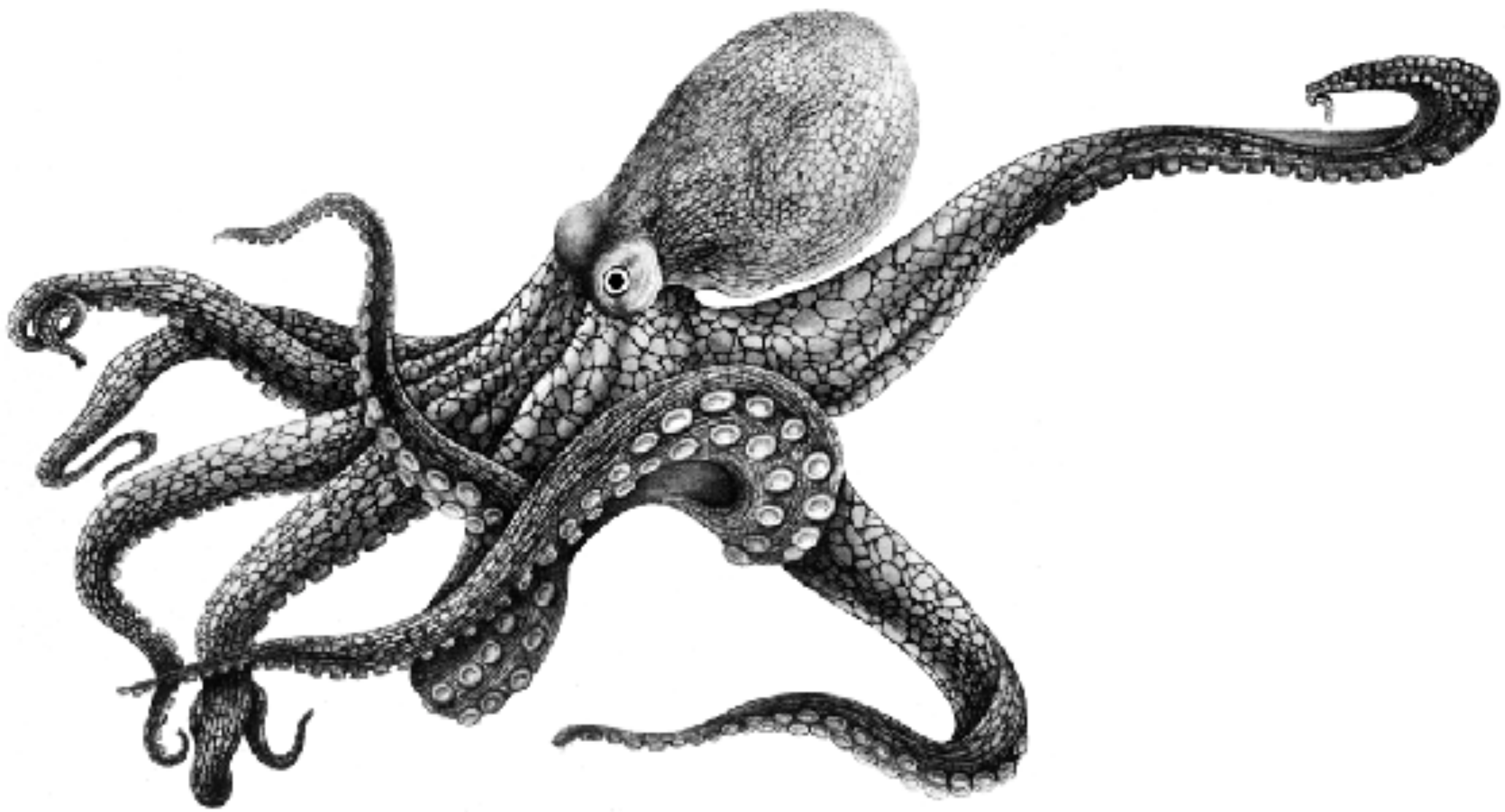

Standard models in IBNR models

- Chain Ladder $C_{i,j+1} = \lambda_j \cdot C_{i,j}$
- Factor models and GLM's (ODP-bootstrap), $\mathbb{E}(Y_{i,j}|\mathcal{F}) = \varphi(A_i, B_j)$
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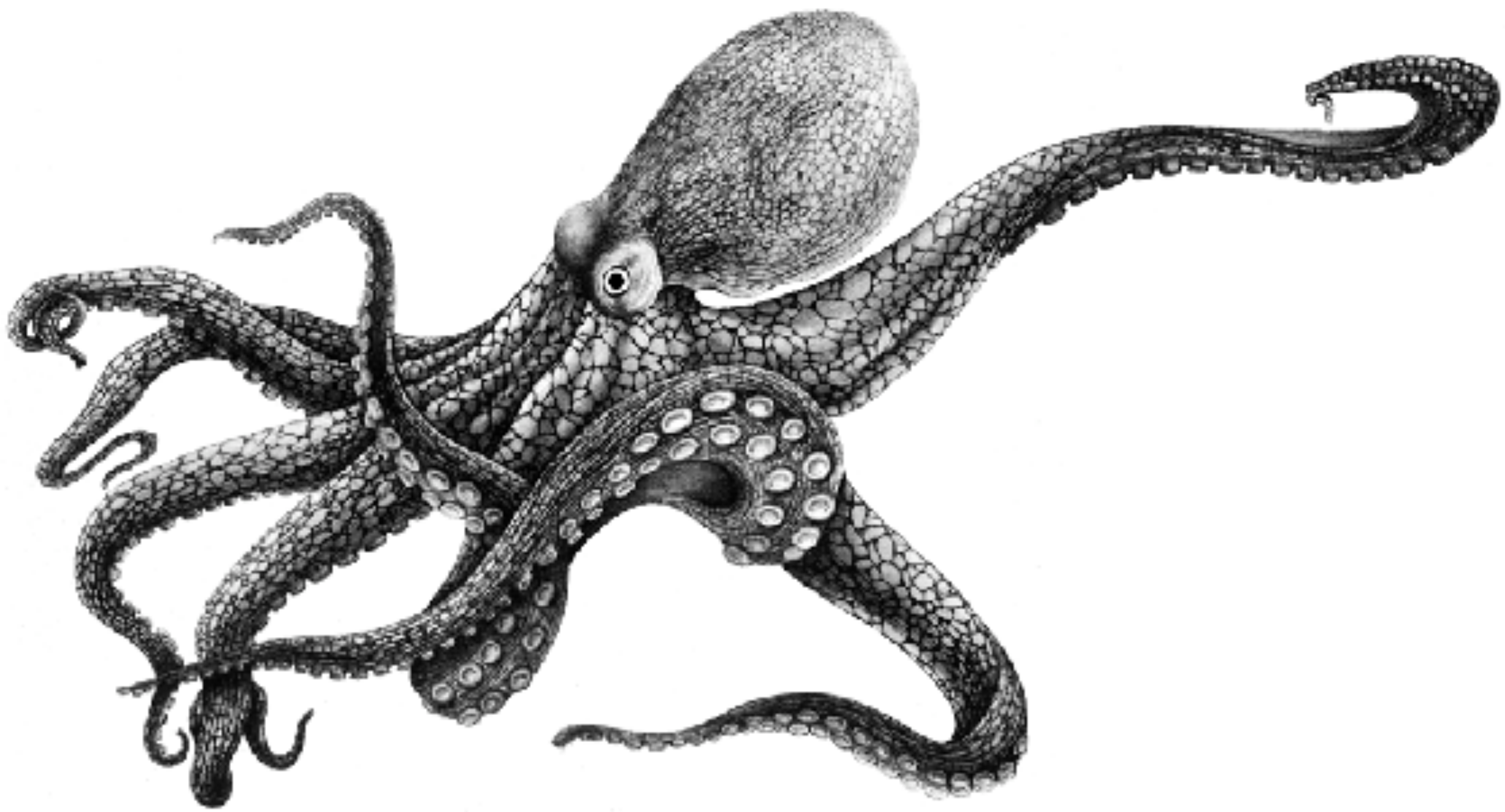

Standard models in IBNR models

- Chain Ladder and Mack (1993) $\mathbb{E}(C_{i,j+1}|\mathcal{F}) = \lambda_j \cdot C_{i,j}$
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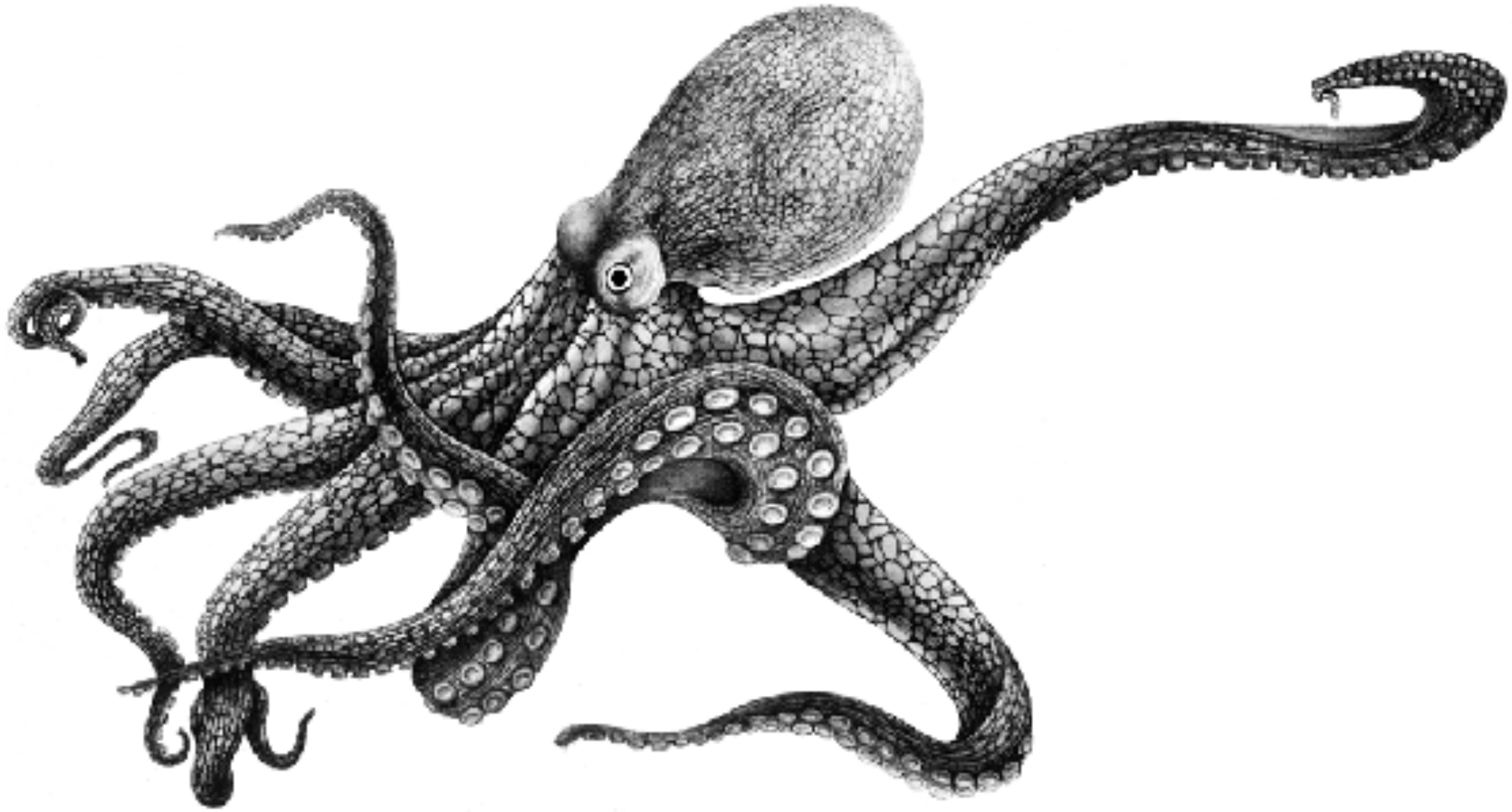

Notations for triangle type data

- $X_{i,j}$ denotes incremental payments, with delay j, for claims occurred year i,
- $C_{i,j}$ denotes cumulated payments, with delay j, for claims occurred year i, $C_{i,j} = X_{i,0} + X_{i,1} + \cdots + X_{i,j}$,

	0	1	2	3	4	5
0	3209	1163	39	17	7	21
1	3367	1292	37	24	10	
2	3871	1474	53	22		
3	4239	1678	103			
4	4929	1865				
5	5217					

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420		
3	4239	5917	6020			
4	4929	6794				
5	5217					

- $\mathcal{F}_t$ denotes information available at time t,

$$\mathcal{F}_t = \{(C_{i,j}), 0 \leq i+j \leq t\} = \{(X_{i,j}), 0 \leq i+j \leq t\}$$

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3	4239	5917	6020			
4	4929	6794				
5	5217					

- ${\mathcal{F}_t}^k$ denotes partial information available at time t, based on the first k years, only

$$ {\mathcal{F}_t}^k = \{(C_{i,j}), 0 \leq i+j \leq t, i \leq k\} = \{(X_{i,j}), 0 \leq i+j \leq t, i \leq k\} $$

Chain Ladder estimation

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	
2	3871	5345	5398	5420		
3	4239	5917	6020			
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5	5217					

	0	1	2	3	4	5
0	3209	4372	4411	4428	4435	4456
1	3367	4659	4696	4720	4730	4752.4
2	3871	5345	5398	5420	5430.1	5455.8
3	4239	5917	6020	6046.15	6057.4	6086.1
4	4929	6794	6871.7	6901.5	6914.3	6947.1
5	5217	7204.3	7286.7	7318.3	7331.9	7366.7

with the following link ratios

	0	1	2	3	4	n
λ_j	1,38093	1,01143	1,00434	1,00186	1,00474	1,0000

One the triangle has been completed, we obtain the amount of reserves, with respectively 22, 36, 66, 153 and 2150 per accident year, i.e. the total is 2427.

How to quantify uncertainty in Solvecny II

In Solvency II, uncertainty is quantified as a dispersion measure (variance or quantile) of changes in prediction, with one year of additional information.

The best estimate at time t is $\widehat{R}_t = \mathbb{E}(C_\infty|\mathcal{F}_t)$ while it become, at time $t+1$ $\widehat{R}_{t+1} = \mathbb{E}(C_\infty|\mathcal{F}_{t+1})$.

The goal is to estimate

$$\mathbb{E}\left([\mathbb{E}(C_\infty|\mathcal{F}_{t+1}) - \mathbb{E}(C_\infty|\mathcal{F}_t)]^2|\mathcal{F}_t\right)$$

Quantifying uncertainty in odds/tails games

In statistics, the mean squared error is a standard measure to quantify the uncertainty of an estimator, i.e.

$$\text{mse}(\widehat{\theta}) = \mathbb{E}\left(\left[\widehat{\theta} - \theta\right]^2\right)$$

In order to formalize the prediction process in claims reserving consider the following simpler case.

Let $\{x_1, \cdots, x_n\}$ denote an i.i.d. $\mathcal{B}(p)$ sample. We want to predict $S_h = X_{n+1} + \cdots + X_{n+h}$. Let ${}_n\widehat{S}_h = \psi(X_{n+1}, \cdots, X_{n+h}) = h \cdot \widehat{p}_n$ denote the *natural* predictor for S_h, at time n.

Since S_h is a random variable (θ was a constant) define

$$\text{mse}({}_n\widehat{S}_h) = \mathbb{E}\left(\left[{}_n\widehat{S}_h - \mathbb{E}(S_h)\right]^2\right)$$

and

$$\text{msep}({}_n\widehat{S}_h) = \mathbb{E}\left(\left[{}_n\widehat{S}_h - S_h\right]^2\right)$$

Note that

$$\begin{aligned}\text{msep}({}_n\widehat{S}_h) &= \mathbb{E}\left(\left[{}_n\widehat{S}_h - \mathbb{E}(S_h)\right]^2\right) + \mathbb{E}\left([\mathbb{E}(S_h) - S_h]^2\right)\\ &= \text{mse}({}_n\widehat{S}_h) + \text{Var}(S_h)\end{aligned}$$

where the first term is a process error and the second term a estimation error.

It is also possible to calculate the information *given the information available* at time n, i.e. a conditional msep,

$$\text{msepc}_n({}_n\widehat{S}_h) = \mathbb{E}\left(\left[{}_n\widehat{S}_h - S_h\right]^2 |\mathcal{F}_n\right)$$

denoted $\mathbb{E}(\text{msepc}_n({}_n\widehat{S}_h)) = \text{msep}({}_n\widehat{S}_h)$.

What are we looking for ?

In Solvency II requirements,

$$CDR_{n+1} = [{}_n\widehat{S}_h] - [x_{n+1} + {}_{n+1}\widehat{S}_{h-1}]$$

This defines a martingale since

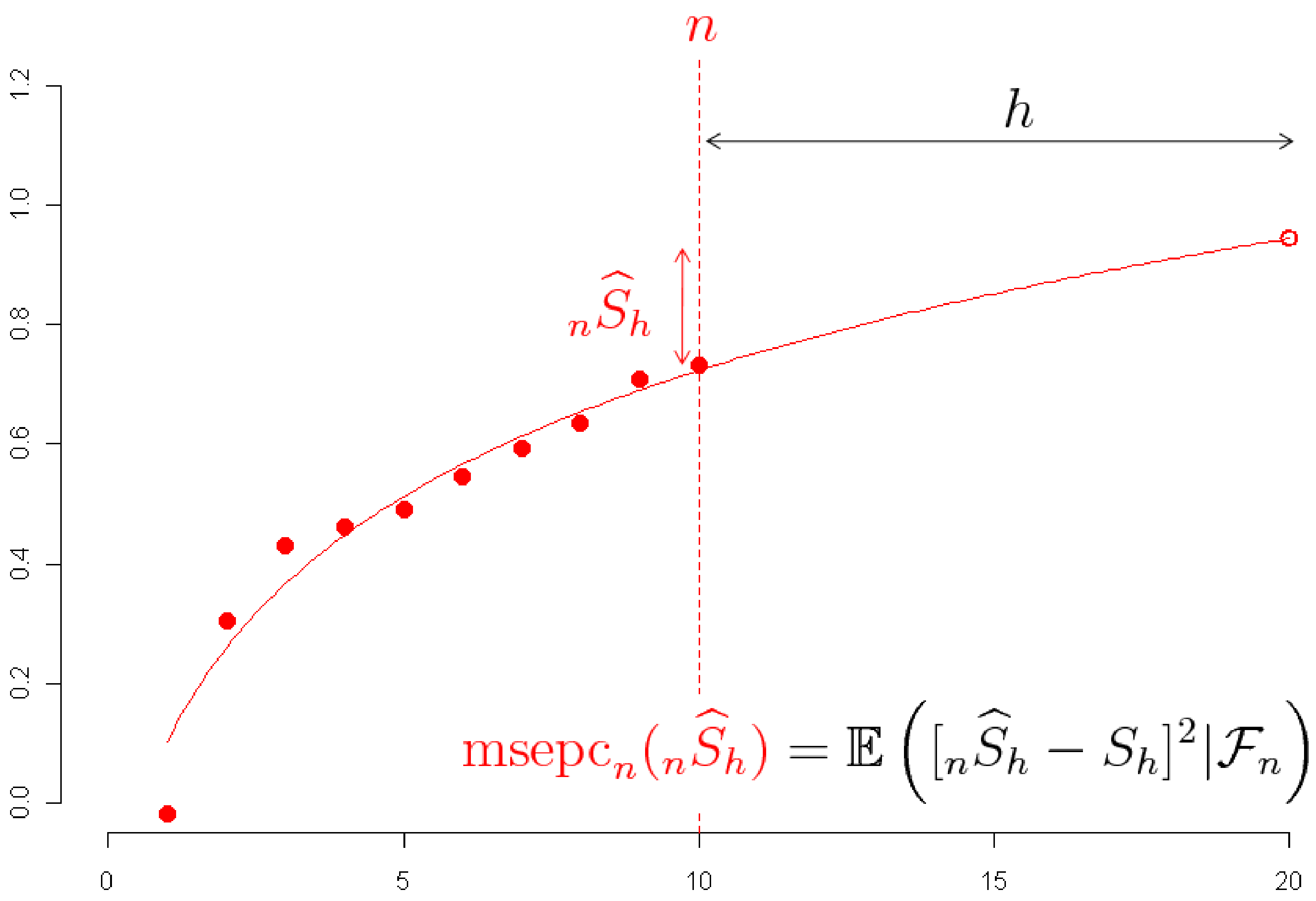
$$\mathbb{E}(CDR_{n+1}|\mathcal{F}_n) = 0$$

and what is required is to estimate

$$\text{msepc}_n(CDR_{n+1})$$

i.e. find $\widehat{\text{msepc}}_n(CDR_{n+1})$.

What are we looking for ?



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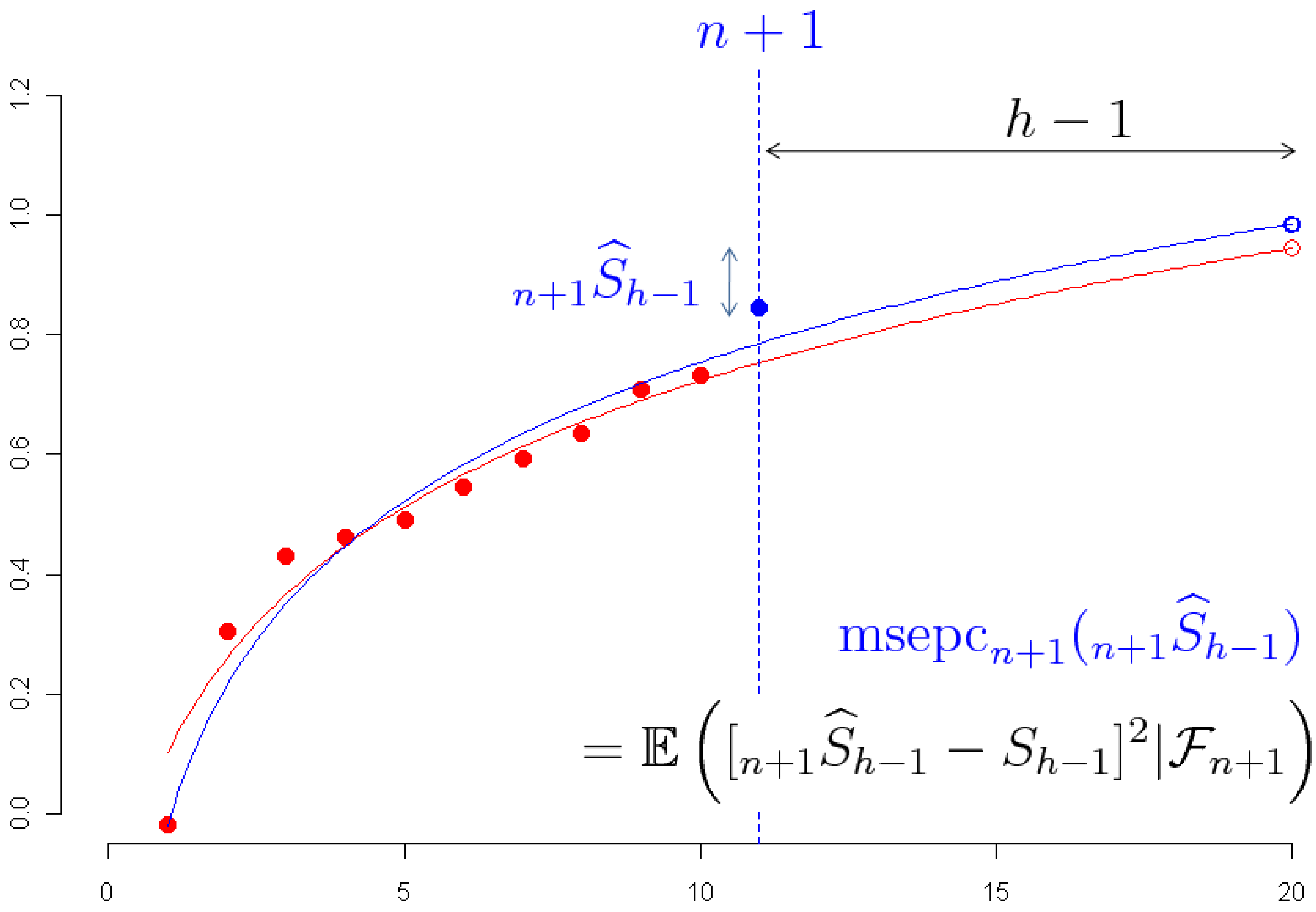
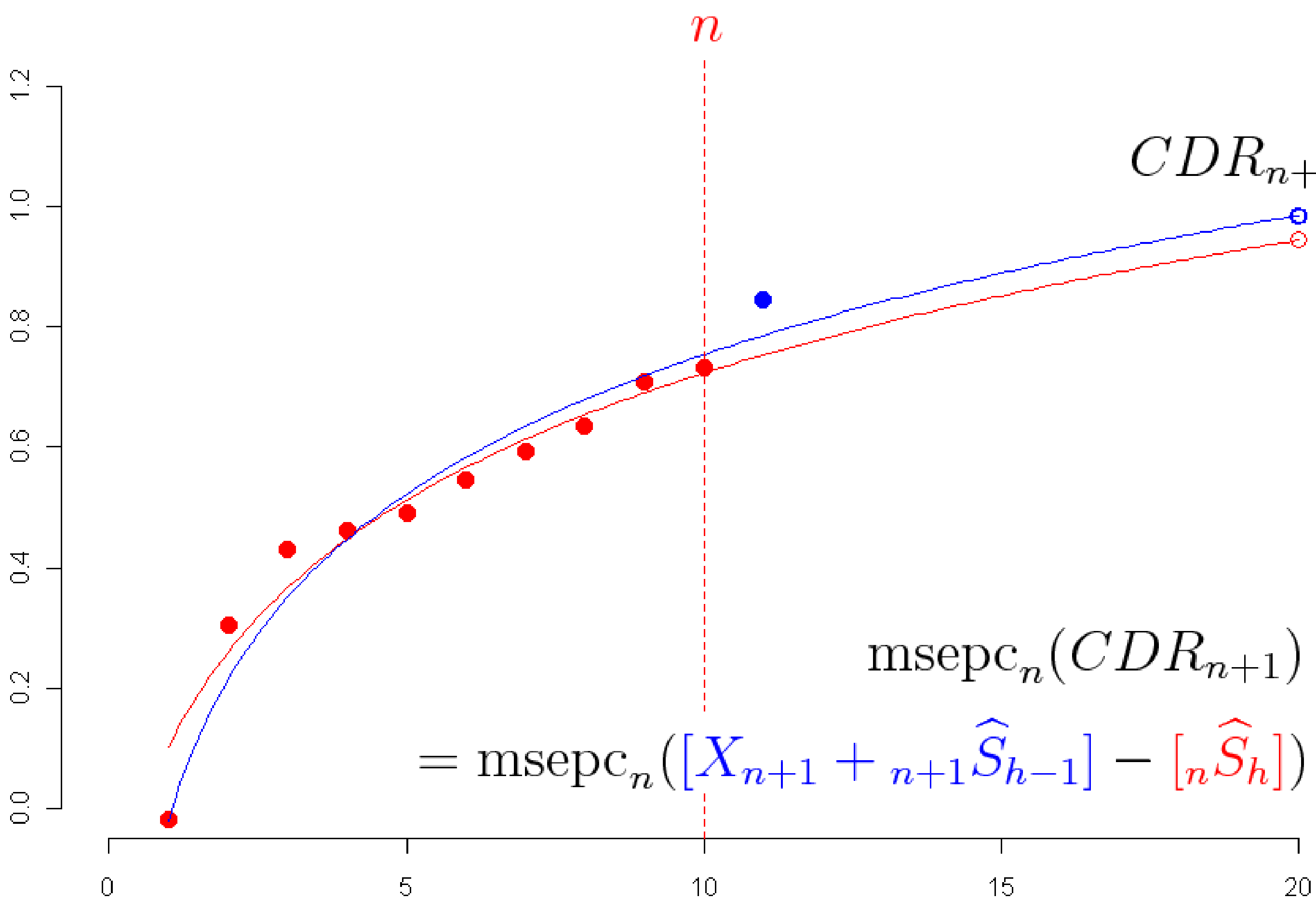


What are we looking for ?



Having an estimator of the uncertainty

Let us continue with our repeated tails/heads game. Let $\widehat{p}_n = [x_1 + \cdots + x_n]/n$, so that

$$\text{Var}(\widehat{p}_n) = \frac{p(1-p)}{n}$$

thus

$$\text{mse}({}_n\widehat{S}_h) = \text{mse}(h \cdot \widehat{p}_n) = h^2 \cdot \text{mse}(\widehat{p}_n) = \frac{h^2}{n} p(1-p),$$

or

$$\text{msep}({}_n\widehat{S}_h) = nhp(1-p) + \frac{h^2}{n} p(1-p) = \frac{nh + h^2}{n} p(1-p)$$

i.e.

$$\text{msep}({}_n\widehat{S}_h) = \frac{h(n+h)}{n} p(1-p).$$

Having an estimator of the uncertainty

Thus, this quantity can be estimated as

$$\widehat{\text{msep}}({}_n\widehat{S}_h) = \frac{h(n+h)}{n}\widehat{p}_n(1-\widehat{p}_n).$$

while the mse estimator was

$$\widehat{\text{mse}}({}_n\widehat{S}_h) = \frac{h^2}{n}\widehat{p}_n(1-\widehat{p}_n)$$

Looking at the msepc at time n, we have

$$\text{msepc}_n({}_n\widehat{S}_h) = \text{Var}(S|\mathcal{F}_n) + \text{mse}({}_n\widehat{S}_h|\mathcal{F}_n)$$

Having an estimator of the uncertainty

Looking at the msepc at time n, we have

$$\text{msepc}_n({}_n\widehat{S}_h) = \text{Var}(S|\mathcal{F}_n) + \text{mse}({}_n\widehat{S}_h|\mathcal{F}_n)$$

where

$$\begin{aligned}\text{Var}(S|\mathcal{F}_n) &= \text{Var}(X_{n+1} + \cdots + X_{n+h}|x_1, \cdots, x_n)\\ &= \text{Var}(X_{n+1} + \cdots + X_{n+h}) = hp(1-p)\end{aligned}$$

and

$$\text{mse}({}_n\widehat{S}_h|\mathcal{F}_n) = \left(\mathbb{E}(S_h|\mathcal{F}_n) - {}_n\widehat{S}_h\right)^2$$

which can be written

$$\text{msepc}_n({}_n\widehat{S}_h) = hp(1-p) + h^2\left(p - \widehat{p}_n\right)^2$$

Having an estimator of the uncertainty

This quantity can be estimated as

$$\widehat{\text{msepc}}_n({}_n\widehat{S}_h) = h\widehat{p}_n(1-\widehat{p}_n) + 0.$$

i.e. we keep only the *variance process* term.

Mack (1993) suggested to use partial information to estimate the second term. Define $D = \{X_i, i \leq n\}$ and $B_k = \{X_i, i \leq n, i \leq k\}$ with $k \leq n$. Define

$$\widehat{\text{msepc}}_n^k({}_n\widehat{S}_h) = h\widehat{p}_n(1-\widehat{p}_n) + h^2\left(\widehat{p}_n - \widehat{p}_k\right)^2$$

The one year horizon uncertainty

In Solvency II, insurance companies are requiered to estimate the msepc, at time n, of the difference between $X_{n+1} + {}_{n+1}\widehat{S}_{(h-1)}$ and ${}_{n}\widehat{S}_{(h)}$.

Those two quantities estimate the same things, at different dates,

- ${}_{n}\widehat{S}_{(h)}$ is a predictor for S_h at time n
- $X_{n+1} + {}_{n+1}\widehat{S}_{(h-1)}$ is a predictor for S_h at time $n+1$,

If we admit that we are looking for the following quantity (as in Merz & Wüthrich (2008))

$$\text{msepc}_n = \mathbb{E}\left([X_{n+1} + (h-1)\cdot \widehat{p}_{n+1} - h\cdot \widehat{p}_n]^2 \,|\mathcal{F}_n \right)$$

i.e.

$$\text{msepc}_n = \mathbb{E}\left(\left[\frac{n+h}{n+1} X_{n+1} + \frac{n-h}{n+1}\widehat{p}_n\right]^2 |\mathcal{F}_n \right)$$

$$\text{msepc}_n = \frac{(n+h)^2}{(n+1)^2} p + \frac{(n+h)(n-h)}{(n+1)^2} p\cdot \widehat{p}_n + \frac{(n-h)^2}{(n+1)^2}\widehat{p}_n^2$$

Updating an estimator, an econometric introduction

If $\widehat{\boldsymbol{\beta}}_n = \left(\boldsymbol{X}'_n\boldsymbol{X}_n\right)^{-1}\boldsymbol{X}'_n\boldsymbol{Y}_n$ denotes the OLS estimate, if a new observation becomes available (y_{n+1}, x_{n+1}), then

$$\widehat{\boldsymbol{\beta}}_{n+1} = \widehat{\boldsymbol{\beta}}_n + \frac{\left(\boldsymbol{X}'_n\boldsymbol{X}_n\right)^{-1} x_{n+1}}{1 + x'_{n+1}\left(\boldsymbol{X}'_n\boldsymbol{X}_n\right)^{-1} x_{n+1}}\left(y_{n+1} - x'_{n+1}\widehat{\boldsymbol{\beta}}_n\right)$$

or

$$\widehat{\boldsymbol{\beta}}_{n+1} = \widehat{\boldsymbol{\beta}}_n + \left(\boldsymbol{X}'_{n+1}\boldsymbol{X}_{n+1}\right)^{-1} x_{n+1}\left(y_{n+1} - x'_{n+1}\widehat{\beta}_n\right)$$

The CDR for a new observation $\boldsymbol{X} = \boldsymbol{x}$ is then

$$CDR_n = \boldsymbol{x}'([\widehat{\boldsymbol{\beta}}_{n+1} - \widehat{\boldsymbol{\beta}}_n]$$

i.e.

$$CDR_n = \boldsymbol{x}'\left(\boldsymbol{X}'_{n+1}\boldsymbol{X}_{n+1}\right)^{-1} x_{n+1}\left(y_{n+1} - x'_{n+1}\widehat{\beta}\right)$$

Mack's ultimate uncertainty

As shown in Mack (1993),

$$\widehat{\text{msep}}(\widehat{R}_i) = \widehat{C}_{i,\infty}^2 \sum_{j=n-i+1}^{n-1} \frac{\widehat{\sigma}_j^2}{\widehat{\lambda}_j^2} \left(\frac{1}{\widehat{C}_{i,j}} + \frac{1}{\widehat{S}_j} \right)$$

where S_j is the sum of cumulated payments on accident years before year $n-j$,

$$S_j = \sum_{i=1}^{n-j} C_{i,j}.$$

Finally, it is possible also to derive an estimator for the aggregate msep (all accident years)

$$\widehat{\text{msep}}(\widehat{R}) = \sum \widehat{\text{msep}}(\widehat{R}_i) + 2\widehat{C}_{i,\infty}^2 \sum_{k=i+1}^{n} \widehat{C}_{k,n} \sum_{j=n-i+1}^{n-1} \frac{\widehat{\sigma}_j^2}{\widehat{\lambda}_j^2 S_j}$$

Mack's ultimate uncertainty

```
> library(ChainLadder)
>  source("http://perso.univ-rennes1.fr/arthur.charpentier/bases.R")
>  MackChainLadder(PAID)
MackChainLadder(Triangle = PAID)
  Latest Dev.To.Date Ultimate    IBNR Mack.S.E CV(IBNR)
1  4,456       1.000    4,456     0.0    0.000      NaN
2  4,730       0.995    4,752    22.4    0.639   0.0285
3  5,420       0.993    5,456    35.8    2.503   0.0699
4  6,020       0.989    6,086    66.1    5.046   0.0764
5  6,794       0.978    6,947   153.1   31.332   0.2047
6  5,217       0.708    7,367 2,149.7   68.449   0.0318
                  Totals
Latest:    32,637.00
Ultimate:  35,063.99
IBNR:       2,426.99
Mack S.E.:     79.30
CV(IBNR):       0.03
```

i.e. $\text{msepc}_6(\widehat{R}) = 79.30$.

Merz & Wüthrich's one year uncertainty

Based on some martingale properties, one can prove that

$$\mathbb{E}(\text{CDR}_i(n+1)|\mathcal{F}_n) = 0$$

(neither boni nor mali can be expected).

Further, it can be proved that $(\text{CDR}_i(n+1))_n$'s are non correlated, and thus

$$\text{msepc}_n(\text{CDR}_i(n+1)) = \text{Var}(\text{CDR}_i(n+1)|\mathcal{F}_n) = \mathbb{E}(\text{CDR}_i(n+1)^2|\mathcal{F}_n)$$

Merz & Wüthrich (2008) proved that the one year horizon error can be estimated with a formula similar to Mack (1993)

$$\widehat{\text{msepc}}_n(\text{CDR}_i(n+1)) = \widehat{C}_{i,\infty}^2 \left(\widehat{\Gamma}_{i,n} + \widehat{\Delta}_{i,n}\right)$$

where

$$\widehat{\Delta}_{i,n} = \frac{\widehat{\sigma}_{n-i+1}^2}{\widehat{\lambda}_{n-i+1}^2 S_{n-i+1}^{n+1}} + \sum_{j=n-i+2}^{n-1} \left(\frac{C_{n-j+1,j}}{S_j^{n+1}}\right)^2 \frac{\widehat{\sigma}_j^2}{\widehat{\lambda}_j^2 S_j^n}$$

and

$$\widehat{\Gamma}_{i,n} = \left(1 + \frac{\widehat{\sigma}^2_{n-i+1}}{\widehat{\lambda}^2_{n-i+1} C_{i,n-i+1}}\right) \prod_{j=n-i+2}^{n-1} \left(1 + \frac{\widehat{\sigma}^2_j}{\widehat{\lambda}^2_j [S_j^{n+1}]^2} C_{n-j+1,j}\right) - 1$$

Merz & Wüthrich (2008) mentioned that this term can be approximated as

$$\widehat{\Gamma}_{i,n} \approx \frac{\widehat{\sigma}^2_{n-i+1}}{\widehat{\lambda}^2_{n-i+1} C_{i,n-i+1}} + \sum_{j=n-i+2}^{n-1} \left(\frac{C_{n-j+1,j}}{S_j^{n+1}}\right)^2 \frac{\widehat{\sigma}^2_j}{\widehat{\lambda}^2_j C_{n-j+1,j}}$$

using a simple development of $\prod(1+u_i) \approx 1 + \sum u_i$, but which is valid *only* if u_i is extremely small, i.e.

$$\frac{\widehat{\sigma}^2_j}{\widehat{\lambda}^2_j} << C_{n-j+1,j}$$

Implementing Merz& Wüthrich's formula

```
> source("http://perso.univ-rennes1.fr/arthur.charpentier/merz-wuthrich-triangle.R")
> MSEP_Mack_MW(PAID,0)
   MSEP Mack MSEP observable approche MSEP observable exacte
1  0.0000000                 0.000000               0.000000
2  0.6393379                 1.424131               1.315292
3  2.5025153                 2.543508               2.543508
4  5.0459004                 4.476698               4.476698
5 31.3319292                30.915407              30.915407
6 68.4489667                60.832875              60.832898
7 79.2954414                72.574735              72.572700
```

Implementing Merz& Wüthrich's formula

Could Merz & Wüthrich's formula end up with more uncertainty than Mack's

```
> Triangle = read.table("http://perso.univ-rennes1.fr/arthur.charpentier/
+            GAV-triangle.csv",sep=";")/1000000
> MSEP_Mack_MW(Triangle,0)
    MSEP Mack MSEP observable approche MSEP observable exacte
1  0.00000000                0.0000000              0.0000000
2  0.01245974                0.1296922              0.1526059
3  0.20943114                0.2141365              0.2144196
4  0.25800338                0.1980723              0.1987730
5  3.05529740                3.0484895              3.0655251
6 58.42939329               57.0561173             67.3757940
7 58.66964613               57.3015524             67.5861066
```

GLM log-Poisson in triangles

Recall that we while to estimate

$$\mathbb{E}([R-\widehat{R}]^2) = \left[\mathbb{E}(R)-\mathbb{E}(\widehat{R})\right]^2 + \text{Var}(R-\widehat{R}) \approx \text{Var}(R) + \text{Var}(\widehat{R})$$

Classically, consider a log-Poisson model, were incremental payments satisfy

$$X_{i,j} \sim \mathcal{P}(\mu_{i,j}) \text{ where } \mu_{i,j} = \exp[\eta_{i,j}] = \exp[\gamma + \alpha_i + \beta_j]$$

Using the delta method, we get that *asymptotically*

$$\text{Var}(\widehat{X}_{i,j}) = \text{Var}(\widehat{\mu}_{i,j}) \approx \left|\frac{\partial \mu_{i,j}}{\partial \eta_{i,j}}\right|^2 \text{Var}(\widehat{\eta}_{i,j})$$

where, since we consider a log link,

$$\frac{\partial \mu_{i,j}}{\partial \eta_{i,j}} = \mu_{i,j}$$

i.e., with an ODP distribution (i.e. $\text{Var}(X_{i,j} = \varphi\mathbb{E}(X_{i,j})$,

$$\mathbb{E}\left([X_{i,j} - \widehat{X}_{i,j}]^2\right) \approx \widehat{\varphi} \cdot \widehat{\mu}_{i,j} + \widehat{\mu}_{i,j}^2 \cdot \widehat{\text{Var}}(\eta_{i,j})$$

and

$$\text{Cov}(X_{i,j}, X_{k,l}) \approx \widehat{\mu}_{i,j} \cdot \widehat{\mu}_{k,l} \cdot \widehat{\text{Cov}}\left(\widehat{\eta}_{i,j}, \widehat{\eta}_{k,l}\right)$$

Thus, since the overall amount of reserves satisfies

$$\mathbb{E}\left([R - \widehat{R}]^2\right) \approx \sum_{i+j-1>n} \widehat{\varphi} \cdot \widehat{\mu}_{i,j} + \widehat{\boldsymbol{\mu}}' \widehat{\text{Var}}(\widehat{\boldsymbol{\eta}})\widehat{\boldsymbol{\mu}}.$$

```
> an <- 6; ligne = rep(1:an, each=an); colonne = rep(1:an, an)
> passe = (ligne + colonne - 1)<=an; np = sum(passe)
> futur = (ligne + colonne - 1)> an; nf = sum(passe)
> INC=PAID
> INC[,2:6]=PAID[,2:6]-PAID[,1:5]
> Y = as.vector(INC)
> lig = as.factor(ligne); col = as.factor(colonne)
> CL <- glm(Y~lig+col, family=quasipoisson)
> Y2=Y; Y2[is.na(Y)]=.001
```

```
> CL2 <- glm(Y2~lig+col, family=quasipoisson)
> YP = predict(CL)
> p = 2*6-1;
> phi.P = sum(residuals(CL,"pearson")^2)/(np-p)
> Sig = vcov(CL)
> X = model.matrix(CL2)
> Cov.eta = X%*%Sig%*%t(X)
> mu.hat = exp(predict(CL,newdata=data.frame(lig,col)))*futur
> pe2 = phi.P * sum(mu.hat) + t(mu.hat) %*% Cov.eta %*% mu.hat
> cat("Total reserve =", sum(mu.hat), "prediction error =", sqrt(pe2),"\n")
Total reserve = 2426.985 prediction error = 131.7726
```

i.e. $\mathbb{E}(\widehat{R} - R) = 131.77.$

Bootstrap and unccertainty

Bootstrap is now a standard nonparametric technique used to quantify uncertainty.

In the linear model, $\widehat{Y}(\boldsymbol{x}) = \mathbb{E}(Y|\boldsymbol{X} = \boldsymbol{x}) = \boldsymbol{x}'\widehat{\boldsymbol{\beta}}$ while $Y(\boldsymbol{x}) = \mathbb{E}(Y|\boldsymbol{X} = \boldsymbol{x}) + \varepsilon$, and the uncertainty is related to

$$\text{Var}(\widehat{Y}(\boldsymbol{x})) = \text{Var}(\boldsymbol{x}'\widehat{\boldsymbol{\beta}}) = \boldsymbol{x}'\text{Var}(\widehat{\boldsymbol{\beta}})\boldsymbol{x}$$

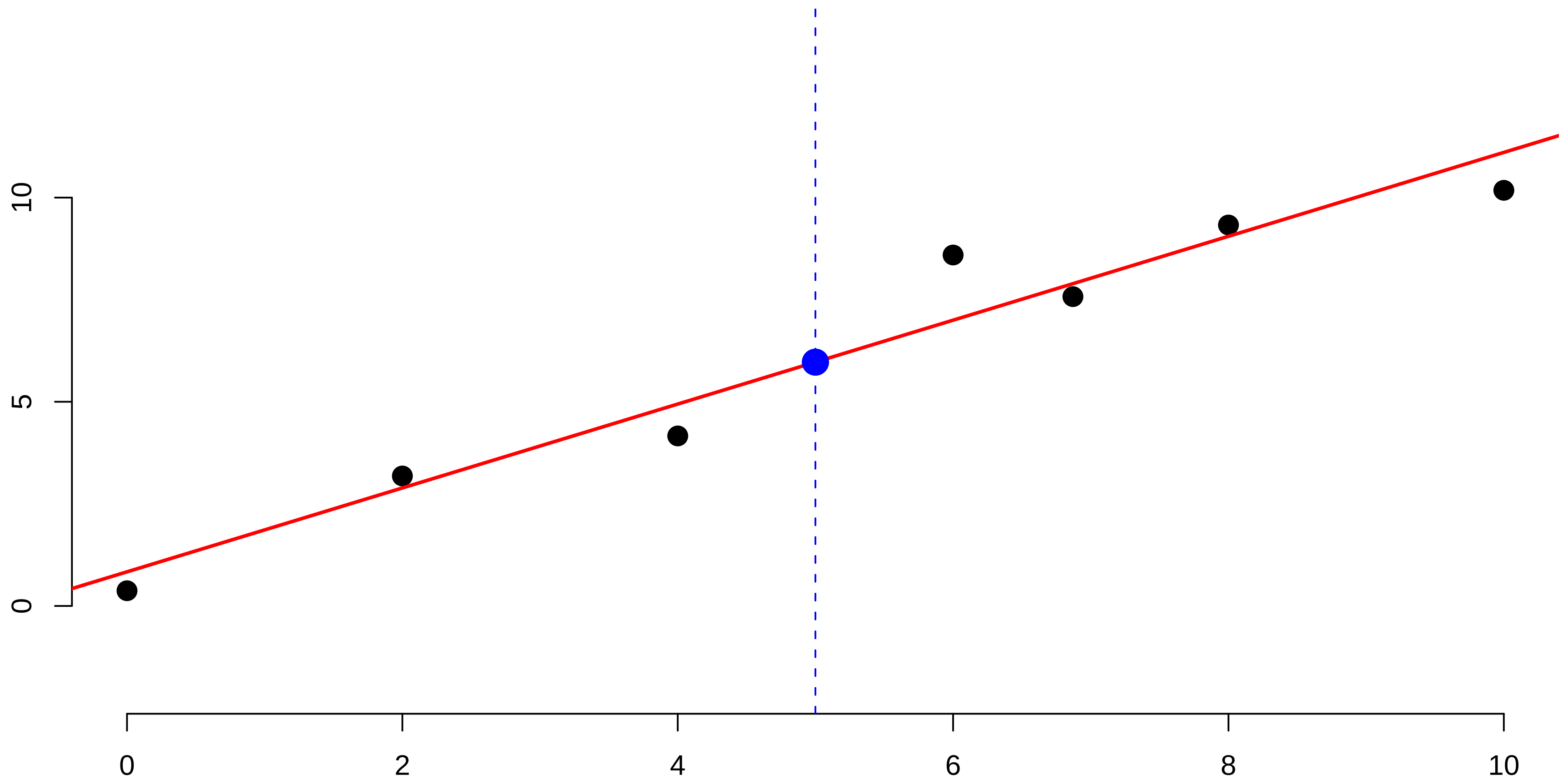
$$\text{Var}(Y(\boldsymbol{x})) = \text{Var}(\boldsymbol{x}'\widehat{\boldsymbol{\beta}} + \widehat{\varepsilon}) \approx \text{Var}(\widehat{Y}(\boldsymbol{x})) + \widehat{\sigma}^2$$

To derive confidence interval or quantiles of $\widehat{Y}(\boldsymbol{x})$ or $Y(\boldsymbol{x})$ we need further assuming, like a distribution for residuals $\widehat{\varepsilon}$

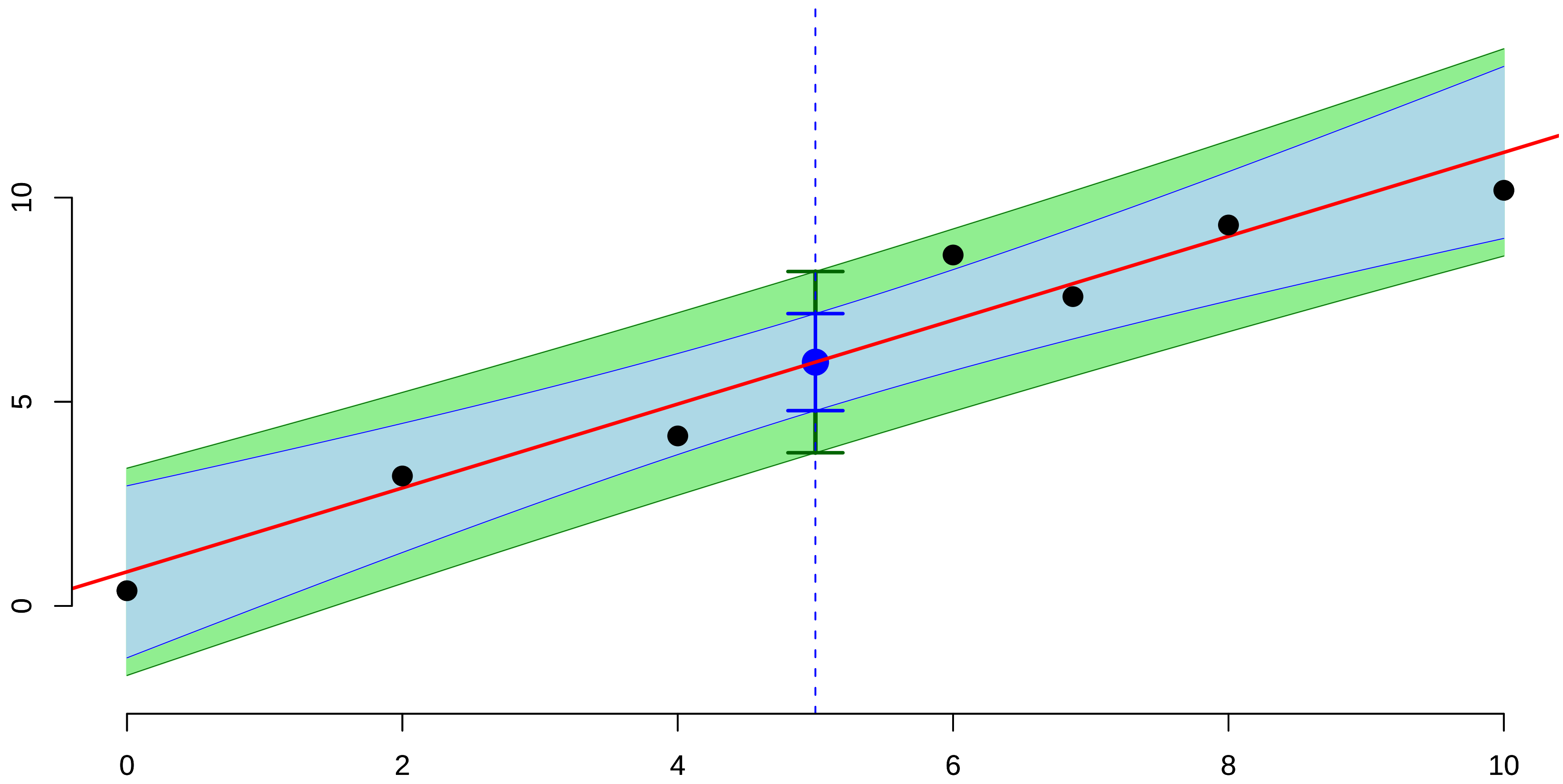
Instead of giving an analytic formula, monte carlo simulations can be used. The idea is to generate samples

$$\{(\boldsymbol{X}_i^\star, Y_i^\star), i = 1, ..., n\} \text{ or } \{(\boldsymbol{X}_i^\star, \widehat{Y}(\boldsymbol{X}_i^\star) + \varepsilon_i^\star), i = 1, ..., n\}$$

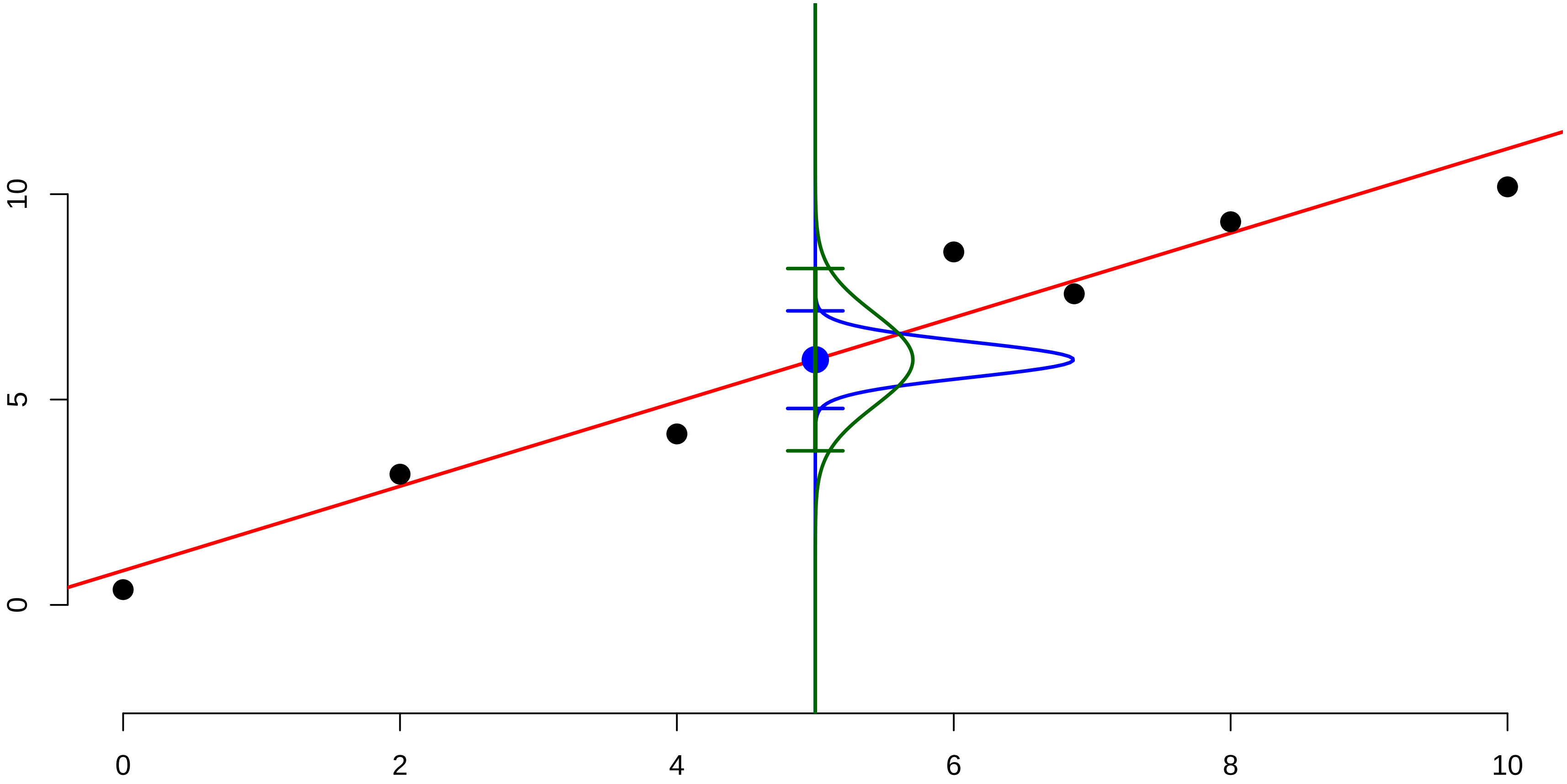
Bootstrap and unccertainty



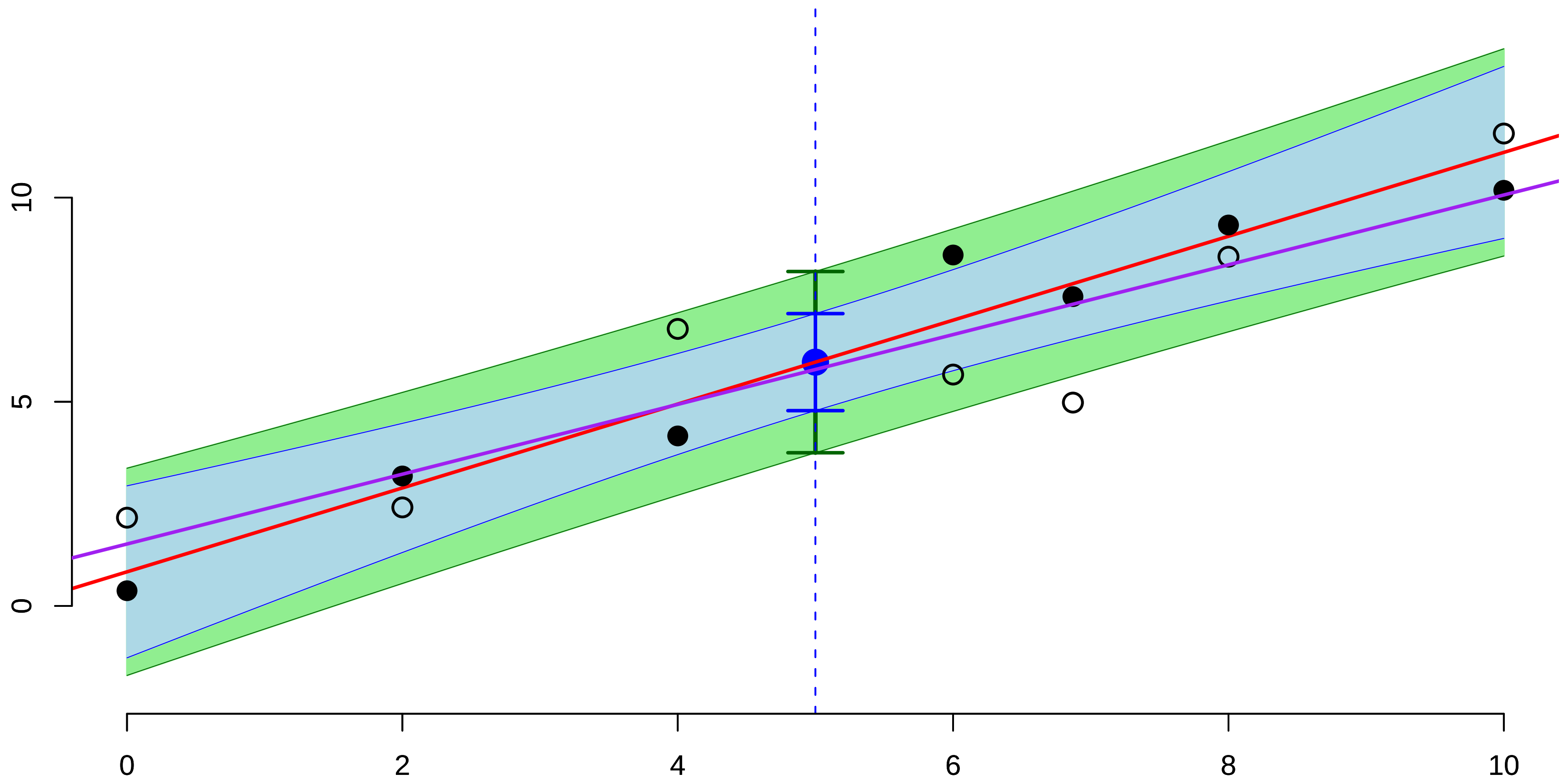
Bootstrap and unccertainty



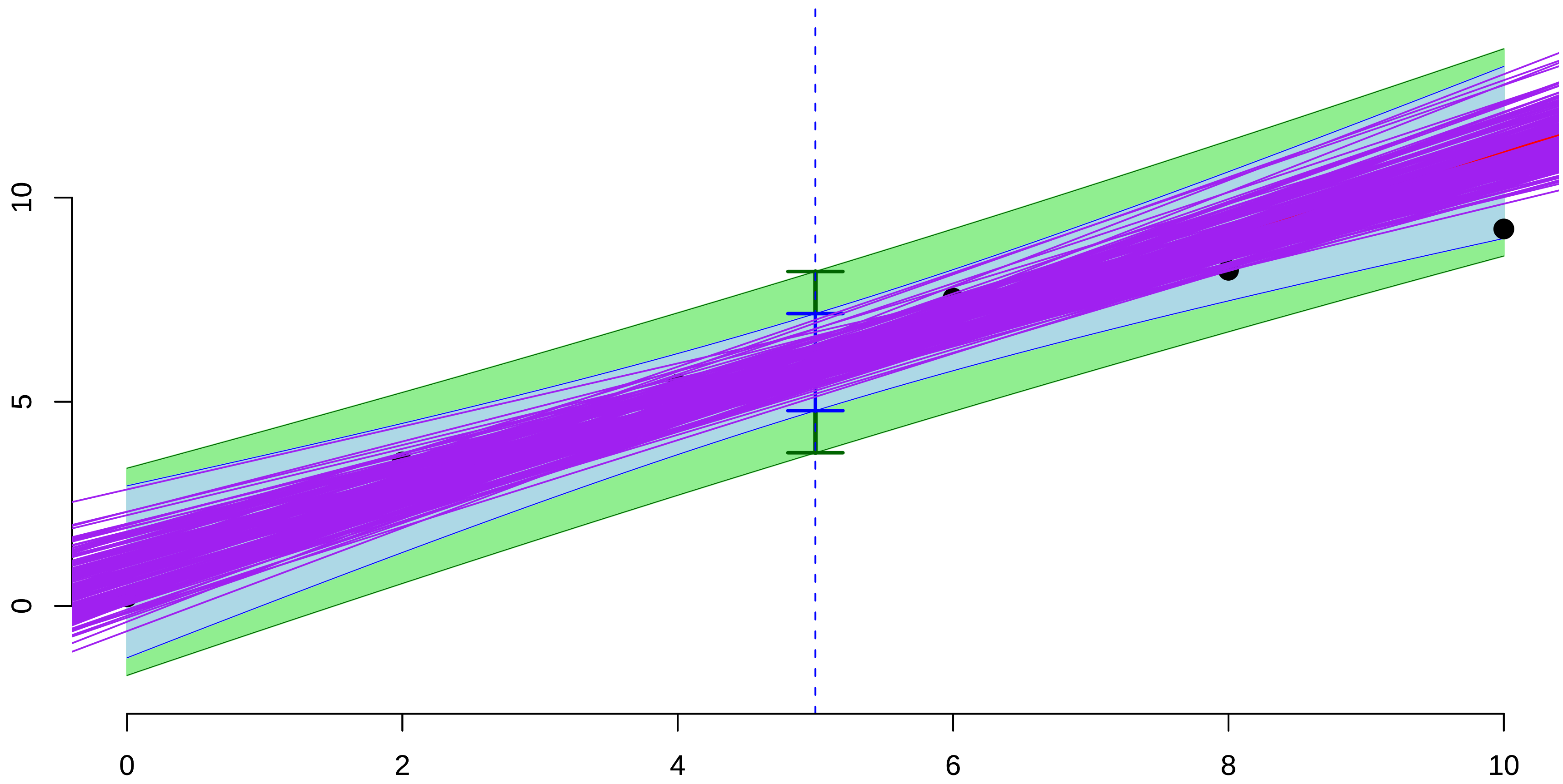
Bootstrap and unccertainty



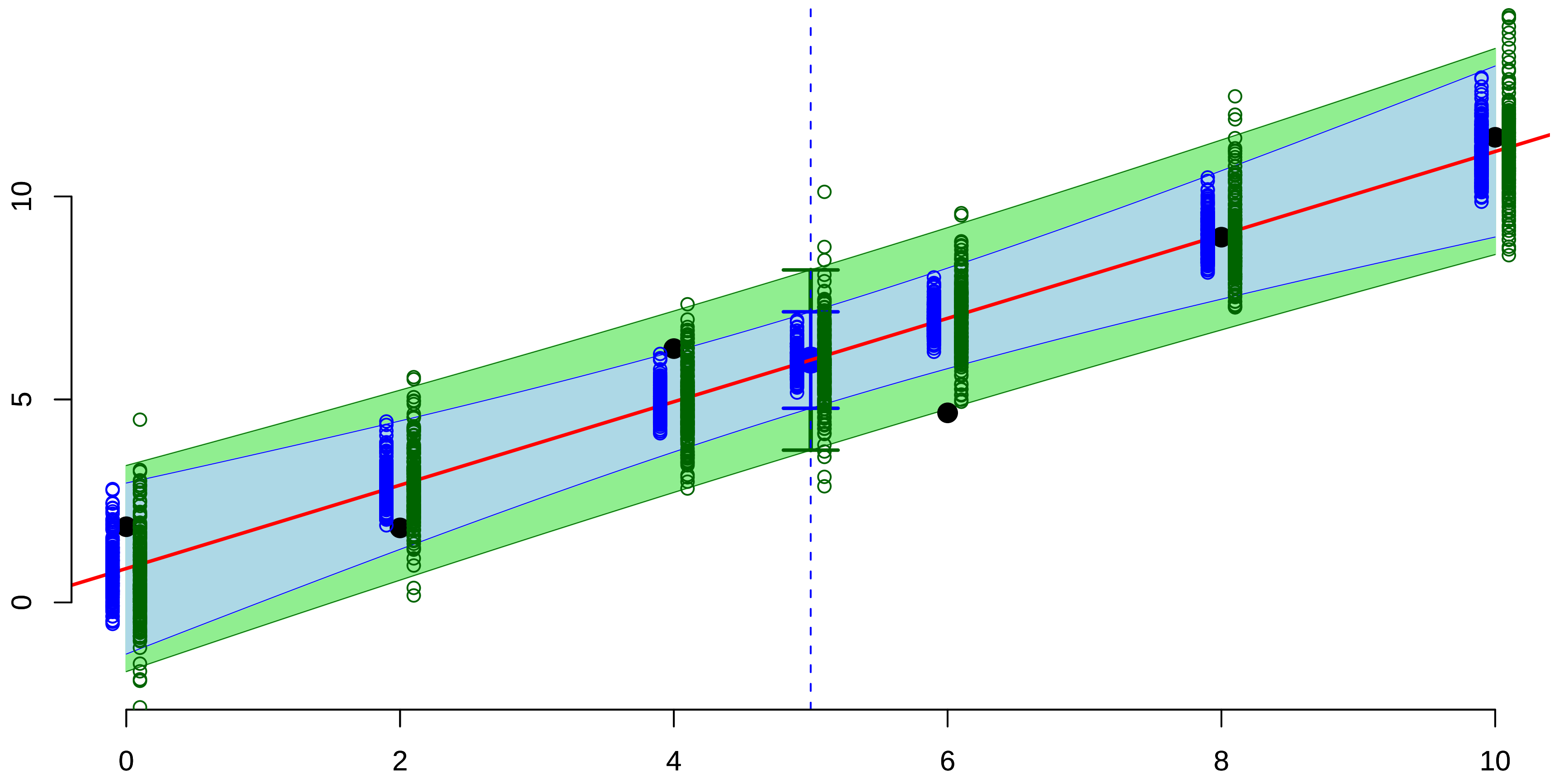
Bootstrap and unccertainty



Bootstrap and unccertainty



Bootstrap and unccertainty



Bootstraping errors ?

Parametric generation : if Z has distribution $F(\cdot)$, then $F^{-1}(\texttt{Random})$ is randomdly distributed according to $F(\cdot)$.

Nonparametric generation : we do not know $F(\cdot)$, it is still possible to estimate it

$$\widehat{F}_n(z) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i \leq x)$$

Then

$$\widehat{F}_n^{-1}(u) = X_{i:n} \text{ where } \frac{i}{n} \leq u < \frac{i+1}{n}$$

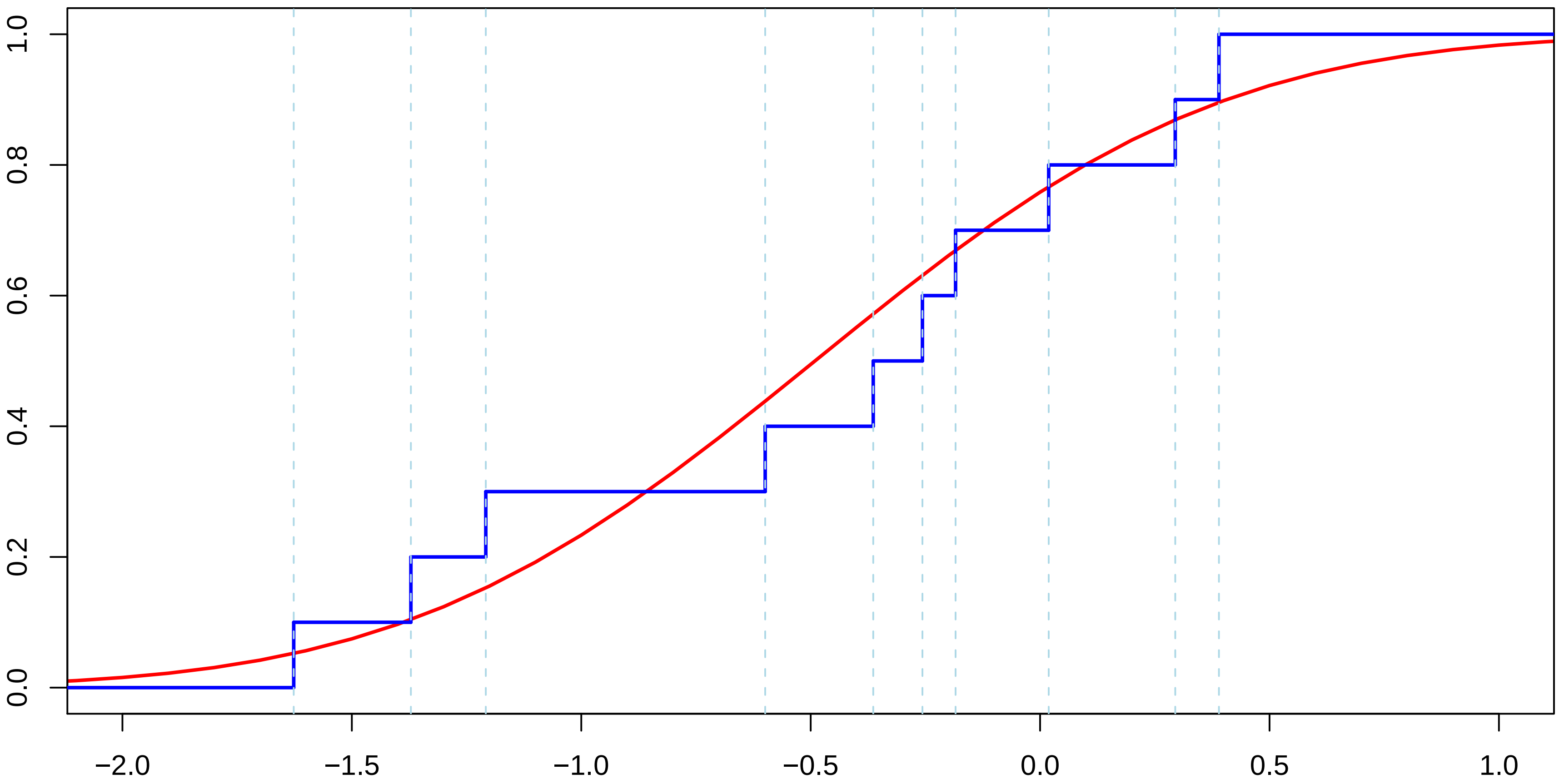
where $X_{i:n}$ denotes the order statistics,

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n-1:n} \leq X_{n:n}$$

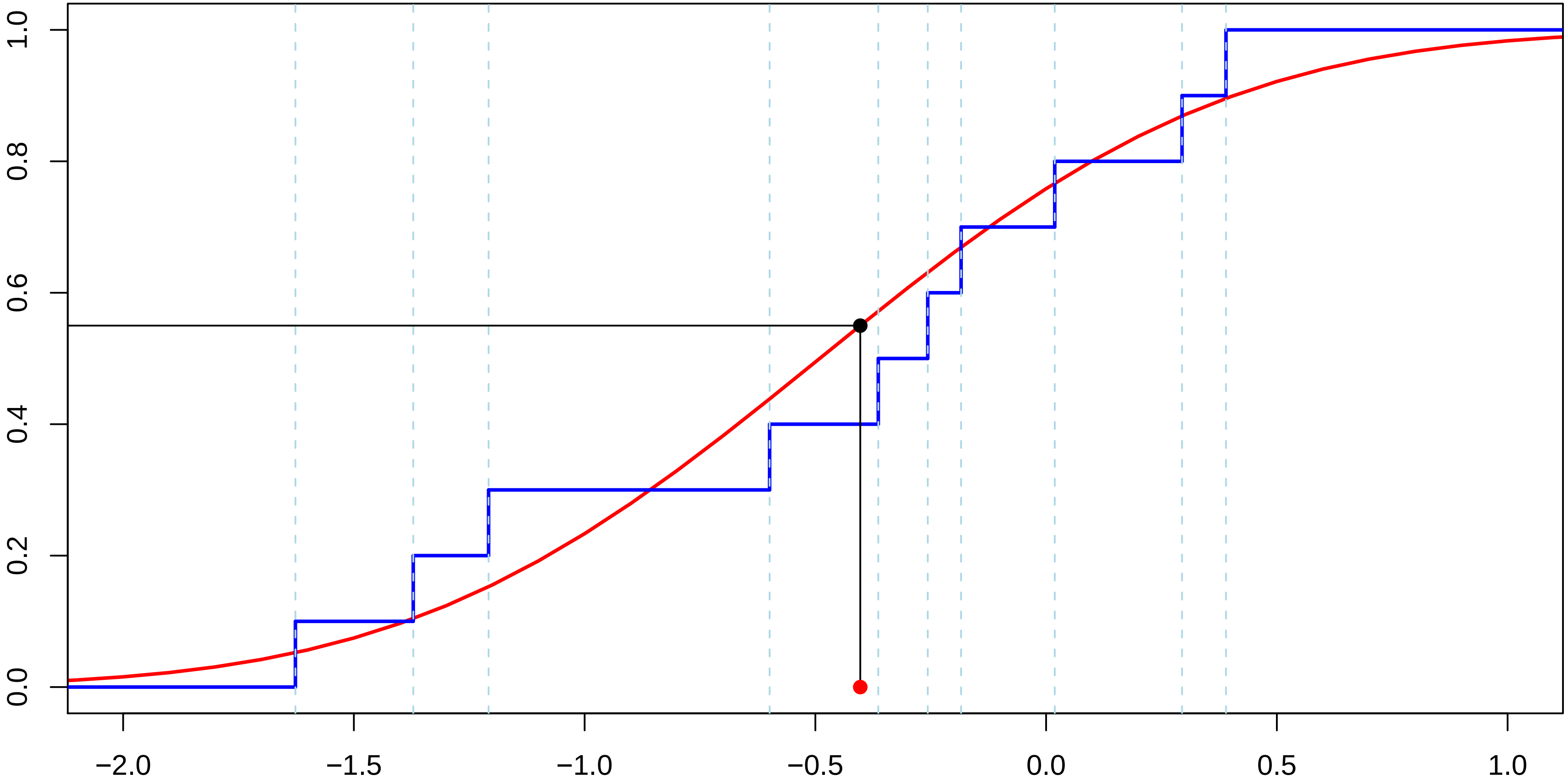
Thus,

$$\widehat{F}_n^{-1}(\texttt{Random}) = X_i \text{ with probability } \frac{1}{n} \text{ for all } i.$$

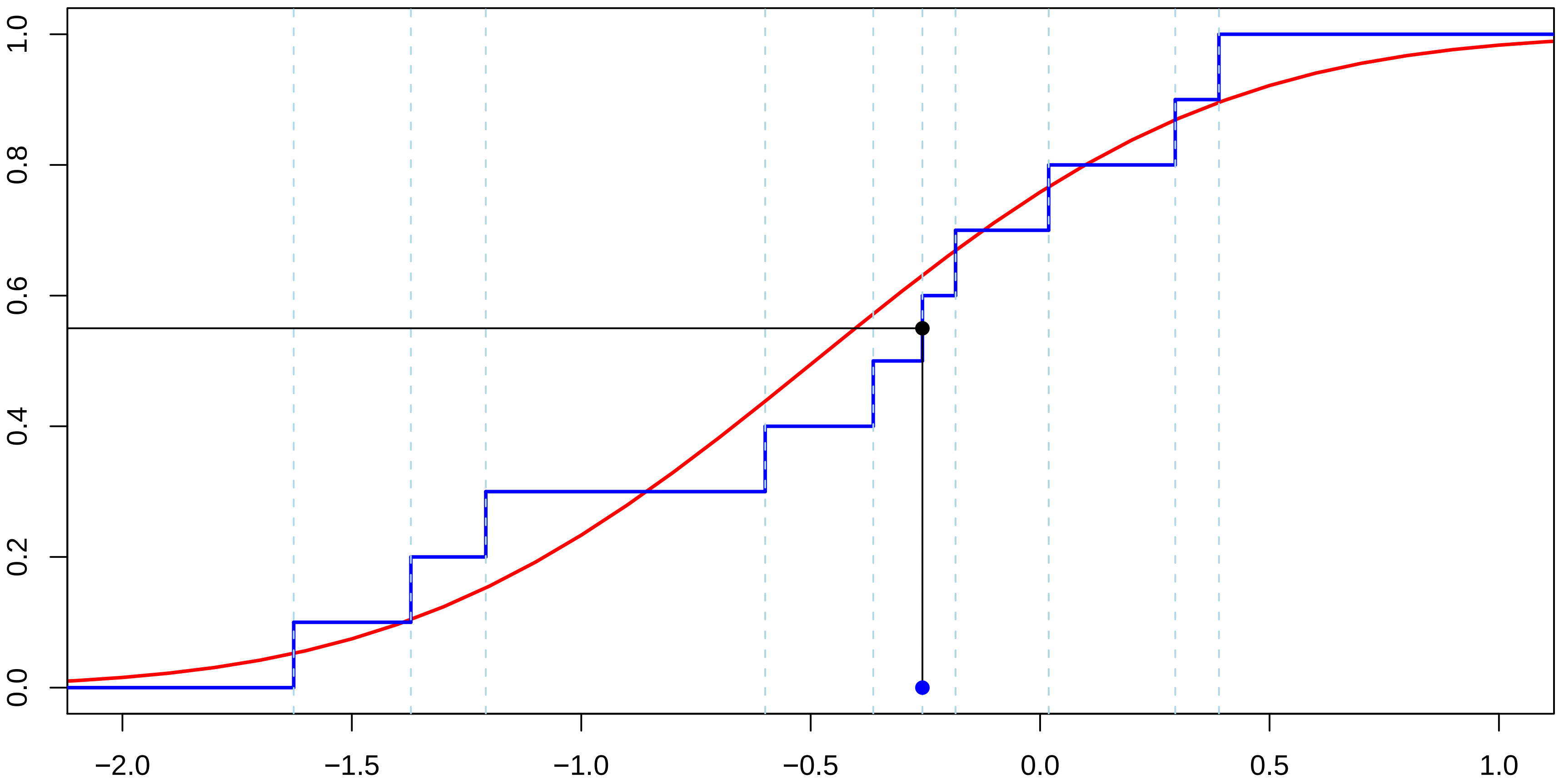
Parametric versus nonparametric random generation



Parametric versus nonparametric random generation



Parametric versus nonparametric random generation



Bootstrap and ultimate uncertainty

From triangle of incremental payments, $(Y_{i,j})$ assume that

$$Y_{i,j} \sim \mathcal{P}(\widehat{Y}_{i,j}) \text{ where } \widehat{Y}_{i,j} = \exp(\widehat{L}_i + \widehat{C}_j)$$

1. Estimate parameters $\widehat{L}_i$ and $\widehat{C}_j$, define Pearson's (pseudo) residuals

$$\widehat{\varepsilon}_{i,j} = \frac{Y_{i,j} - \widehat{Y}_{i,j}}{\sqrt{\widehat{Y}_{i,j}}}$$

2. Generate pseudo triangles on the past, $\{i + j \leq t\}$

$$Y^\star_{i,j} = \widehat{Y}_{i,j} + \widehat{\varepsilon}^\star_{i,j}\sqrt{\widehat{Y}_{i,j}}$$

3. (re)Estimate parameters $\widehat{L}^\star_i$ and $\widehat{C}^\star_j$, and derive expected payments for the future, $\widehat{Y}^\star_{i,j}$.

$$\widehat{R} = \sum_{i+j>t} \widehat{Y}^\star_{i,j}$$

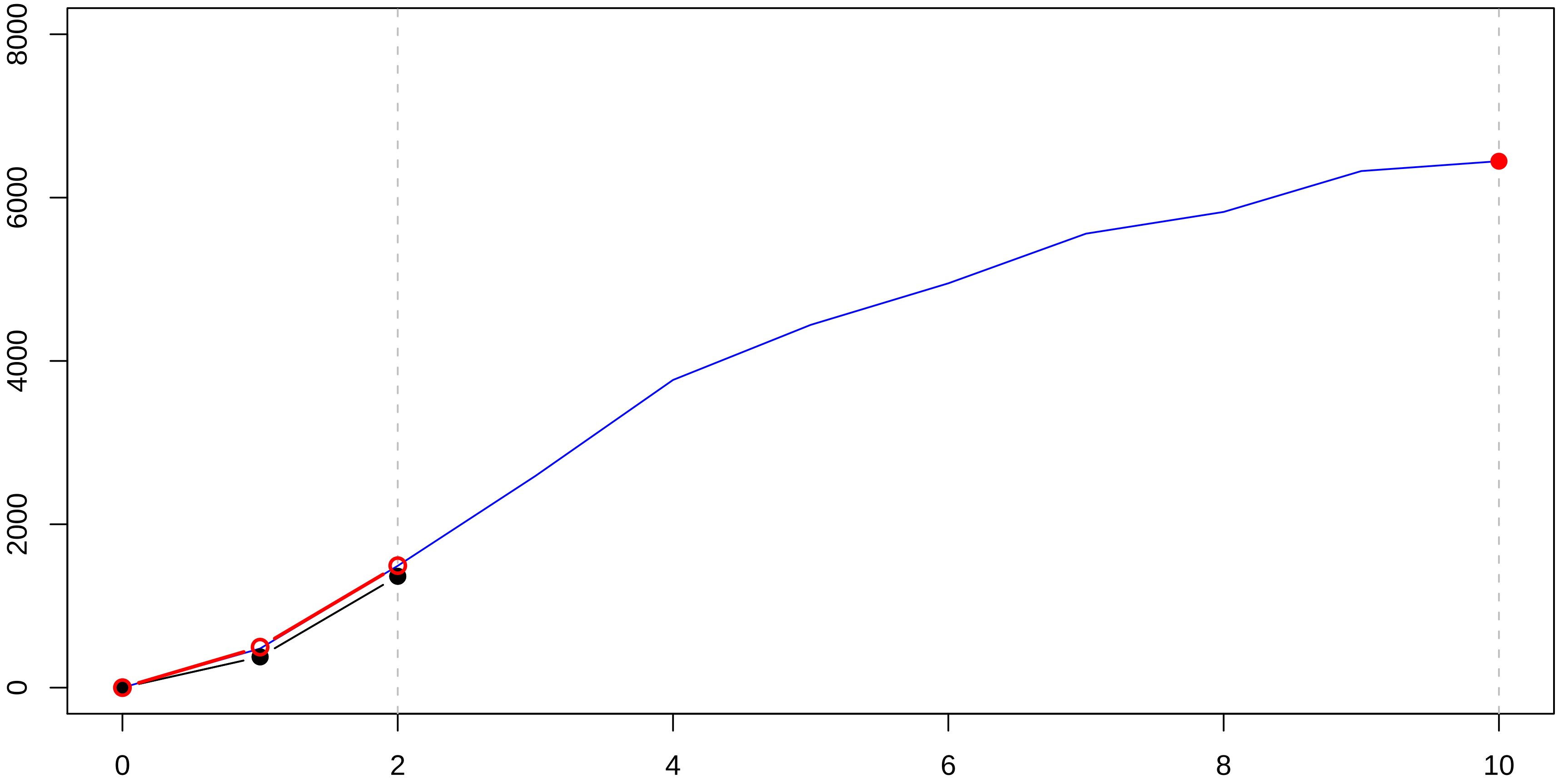
is the best estimate.

4. Generate a scenario for future payments, $Y^\star_{i,j}$ e.g. from a Poisson distribution $\mathcal{P}(\widehat{Y}^\star_{i,j})$

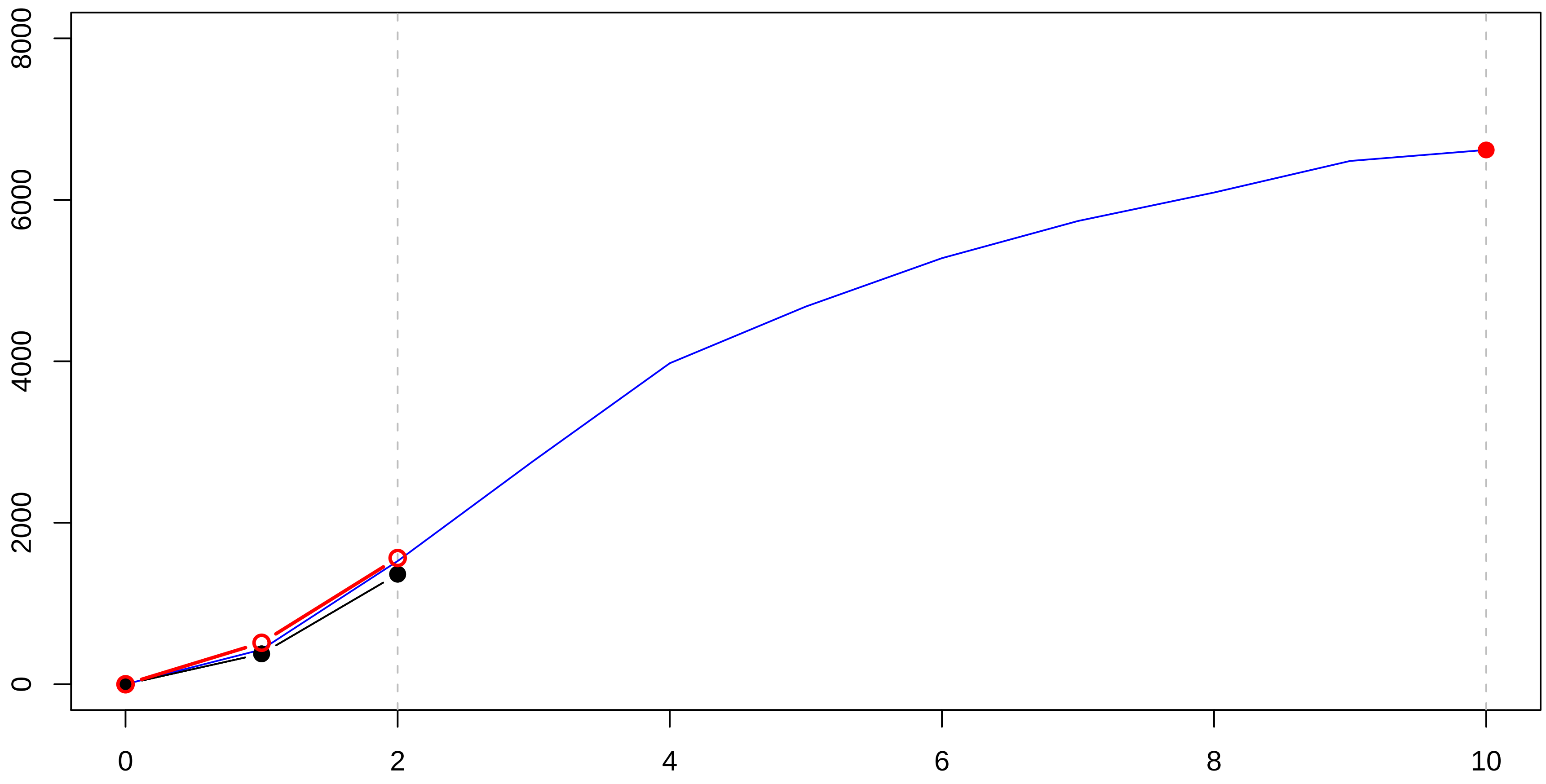
$$R = \sum_{i+j>t} Y^\star_{i,j}$$

One needs to repeat steps 2-4 several times to derive a distribution for R.

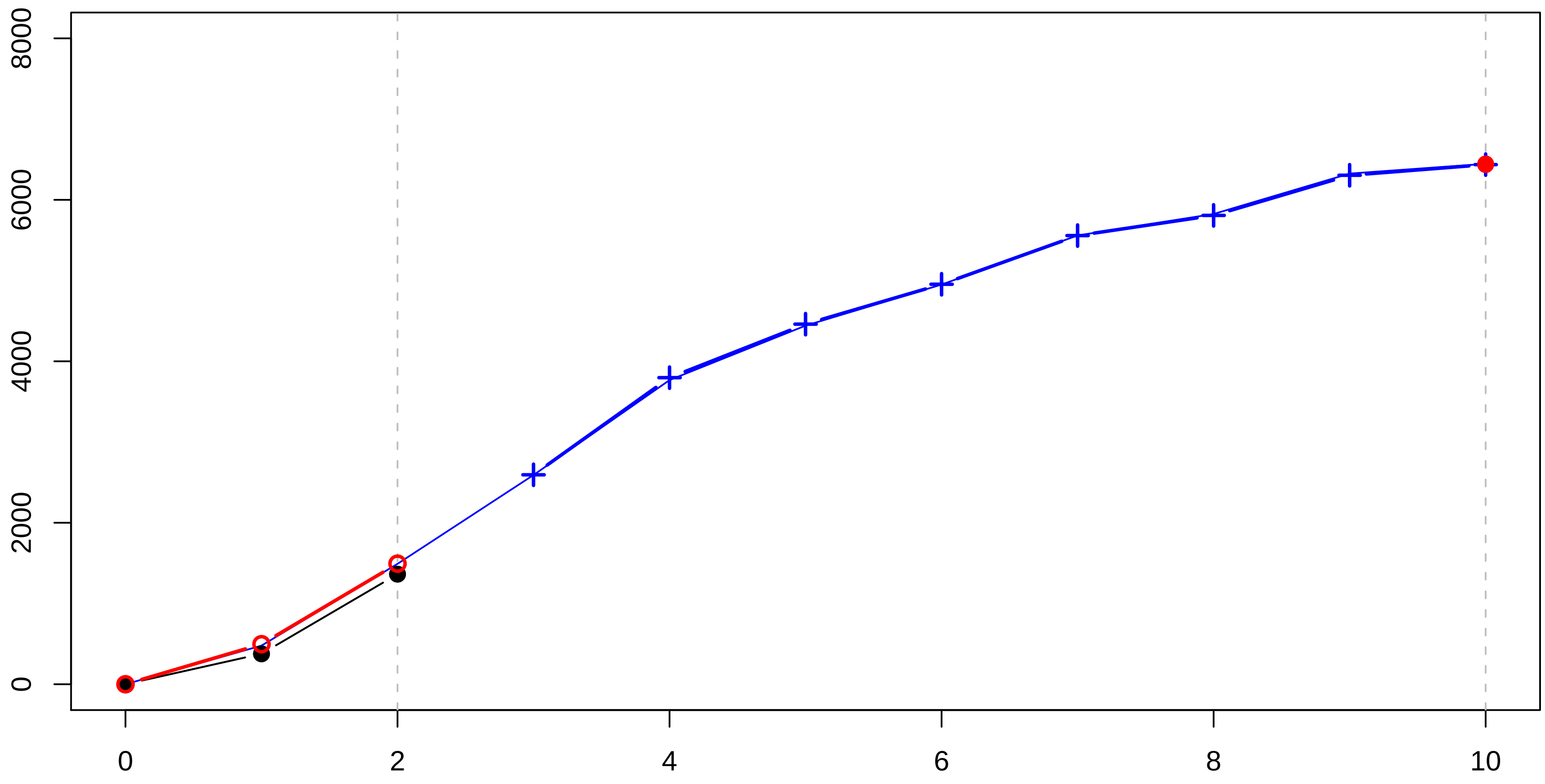
Bootstrap and GLM log-Poisson in triangles



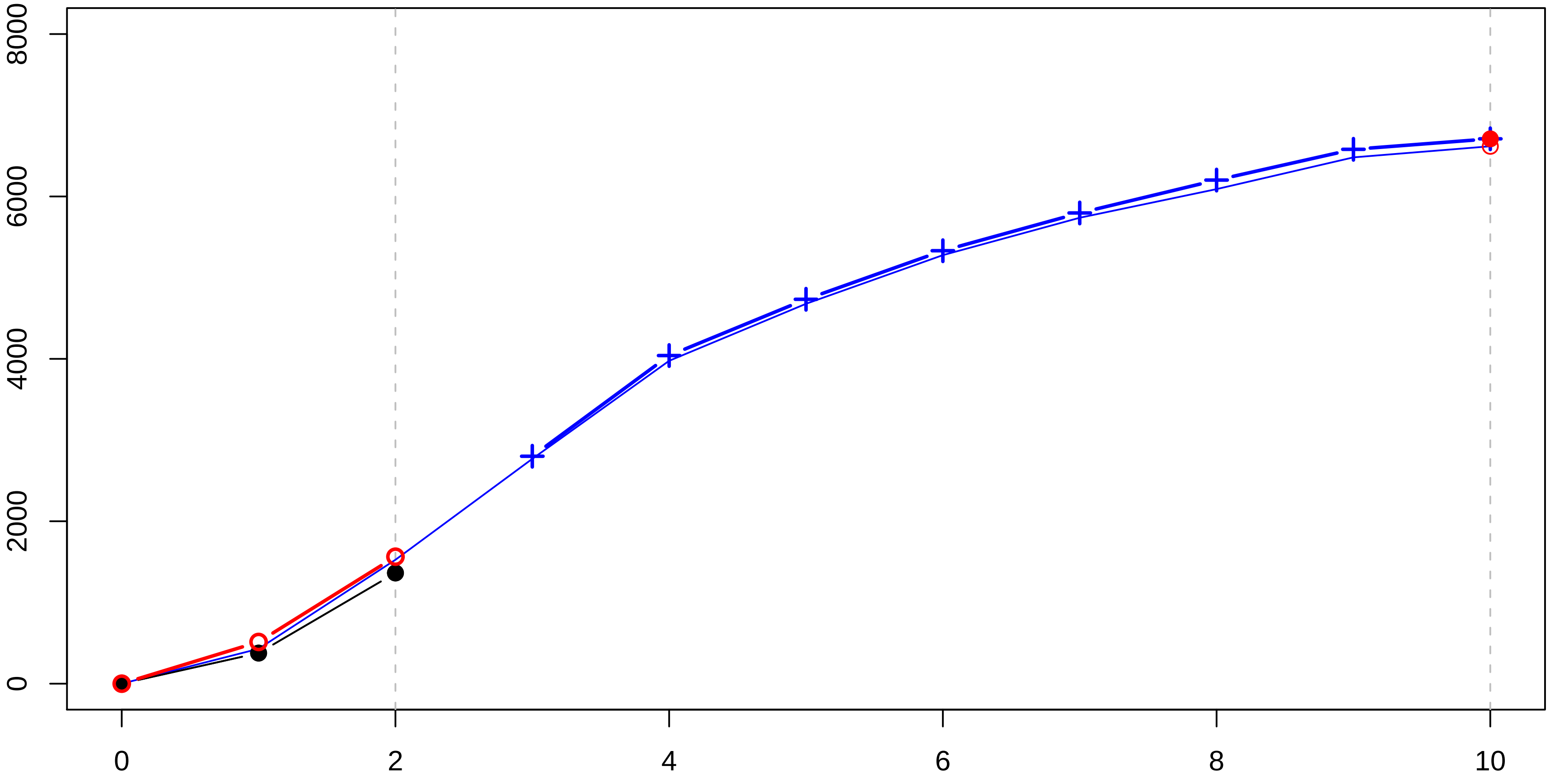
Bootstrap and GLM log-Poisson in triangles



Bootstrap and GLM log-Poisson in triangles

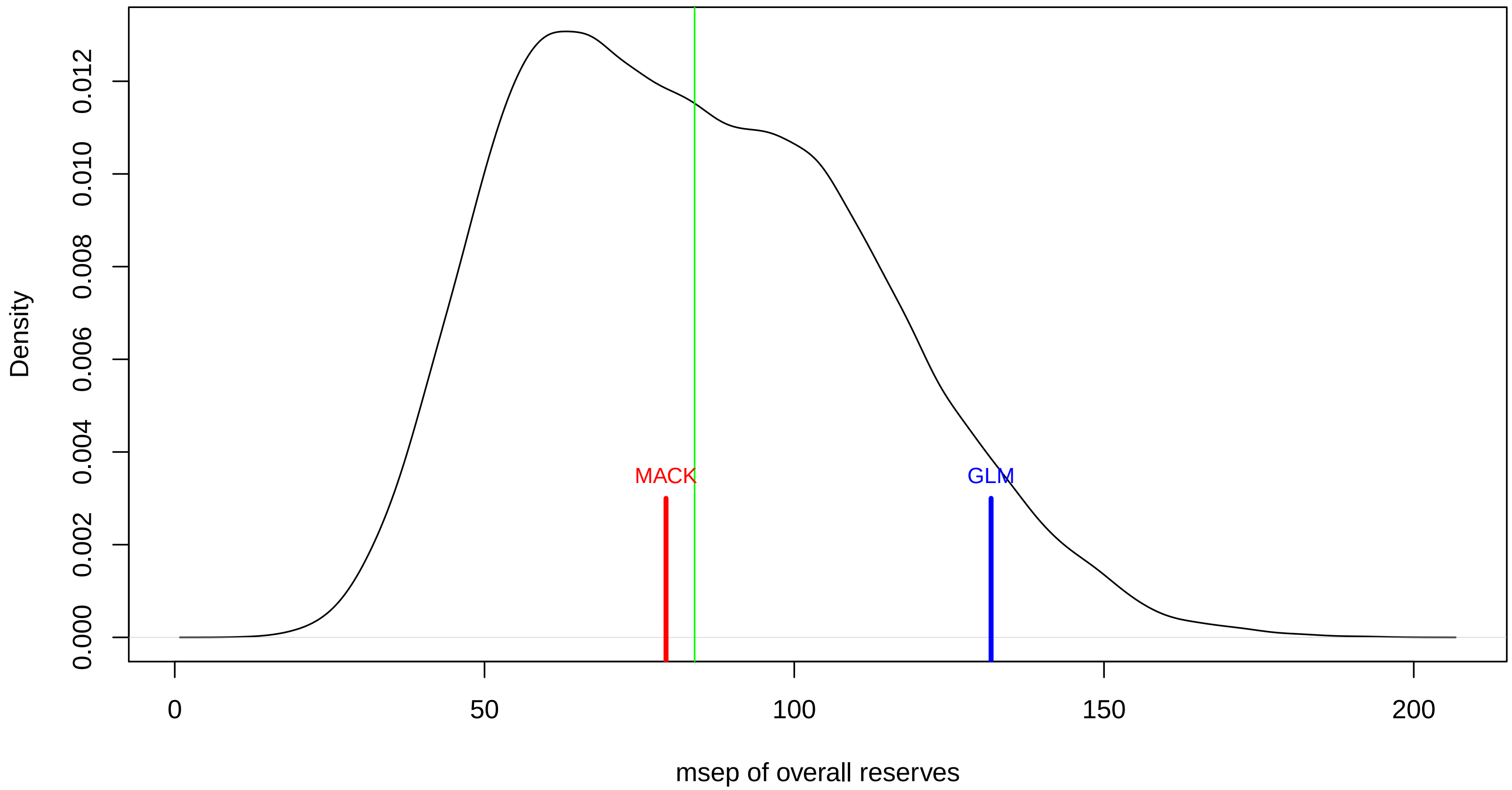


Bootstrap and GLM log-Poisson in triangles



Bootstrap and GLM log-Poisson in triangles

If we repeat it 50,000 times, we obtain the following distribution for the mse.



Bootstrap and one year uncertainty

2. Generate pseudo triangles on the past *and next year* $\{i+j \leq t+1\}$

$$Y^{\star}_{i,j} = \widehat{Y}_{i,j} + \widehat{\varepsilon}^{\star}_{i,j}\sqrt{\widehat{Y}_{i,j}}$$

3. Estimate parameters $\widehat{L}^{\star}_i$ and $\widehat{C}^{\star}_j$, on the past, $\{i+j \leq t\}$, and derive expected payments for the future, $\widehat{Y}^{\star}_{i,j}$.

$$\widehat{R}_t = \sum_{i+j>t} \widehat{Y}_{i,j}$$

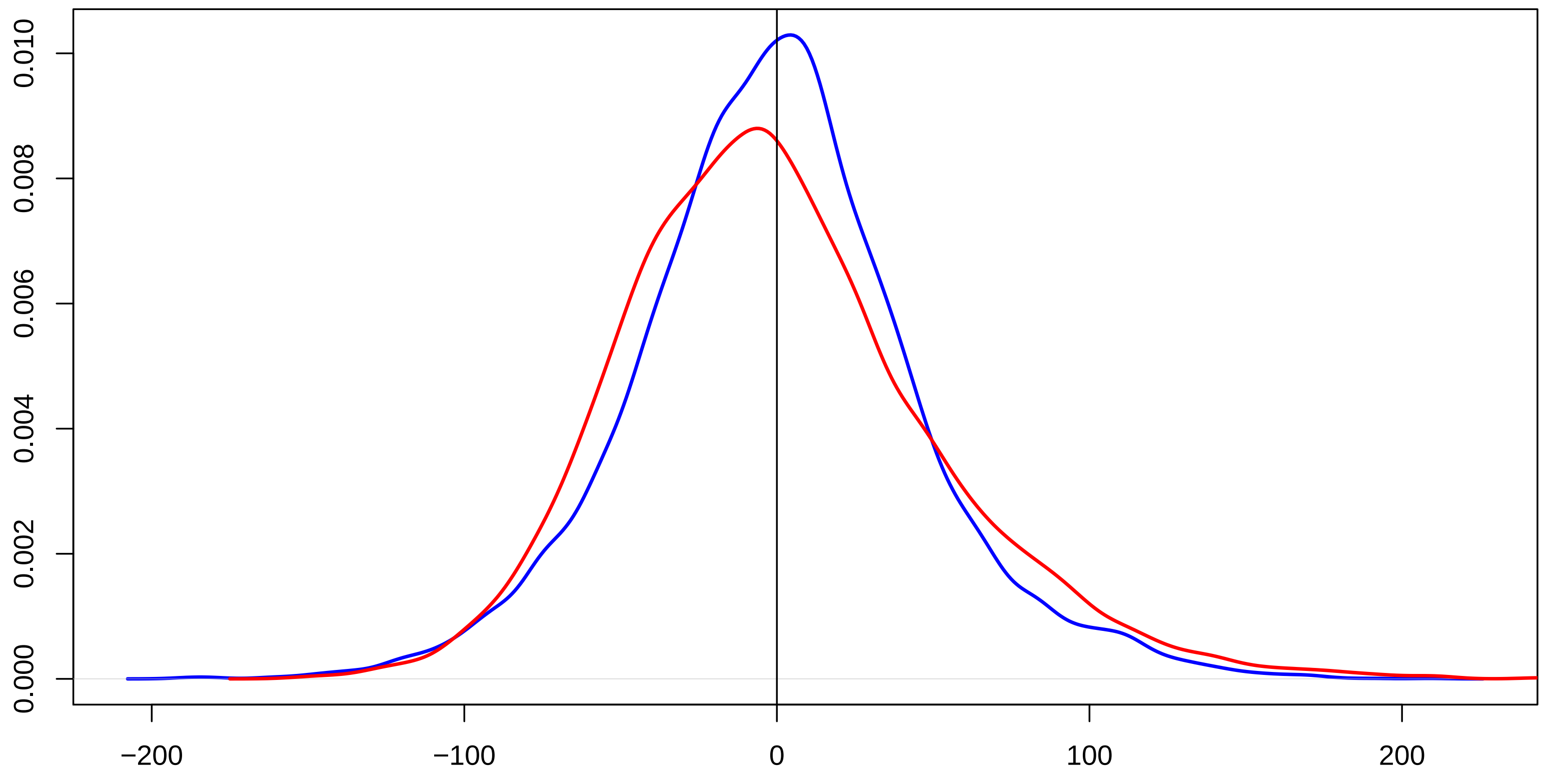
4. Estimate parameters $\widehat{L}^{\star}_i$ and $\widehat{C}^{\star}_j$, on the past and next year, $\{i+j \leq t+1\}$, and derive expected payments for the future, $\widehat{Y}^{\star}_{i,j}$.

$$\widehat{R}_{t+1} = \sum_{i+j>t} \widehat{Y}_{i,j}$$

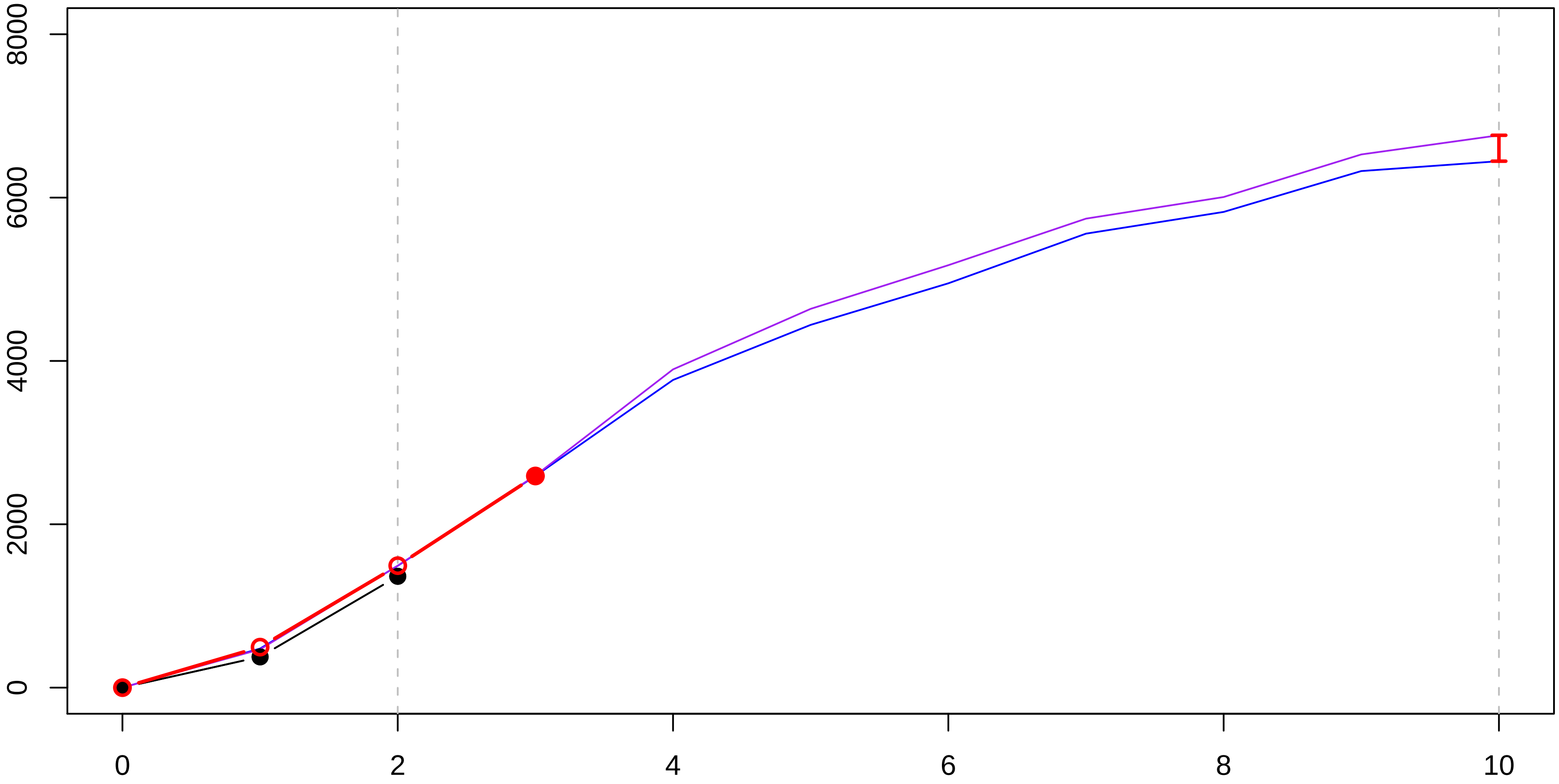
5. Calculate CDR as CDR$=\widehat{R}_{t+1} - \widehat{R}_t$.

Ultimate versus one year uncertainty

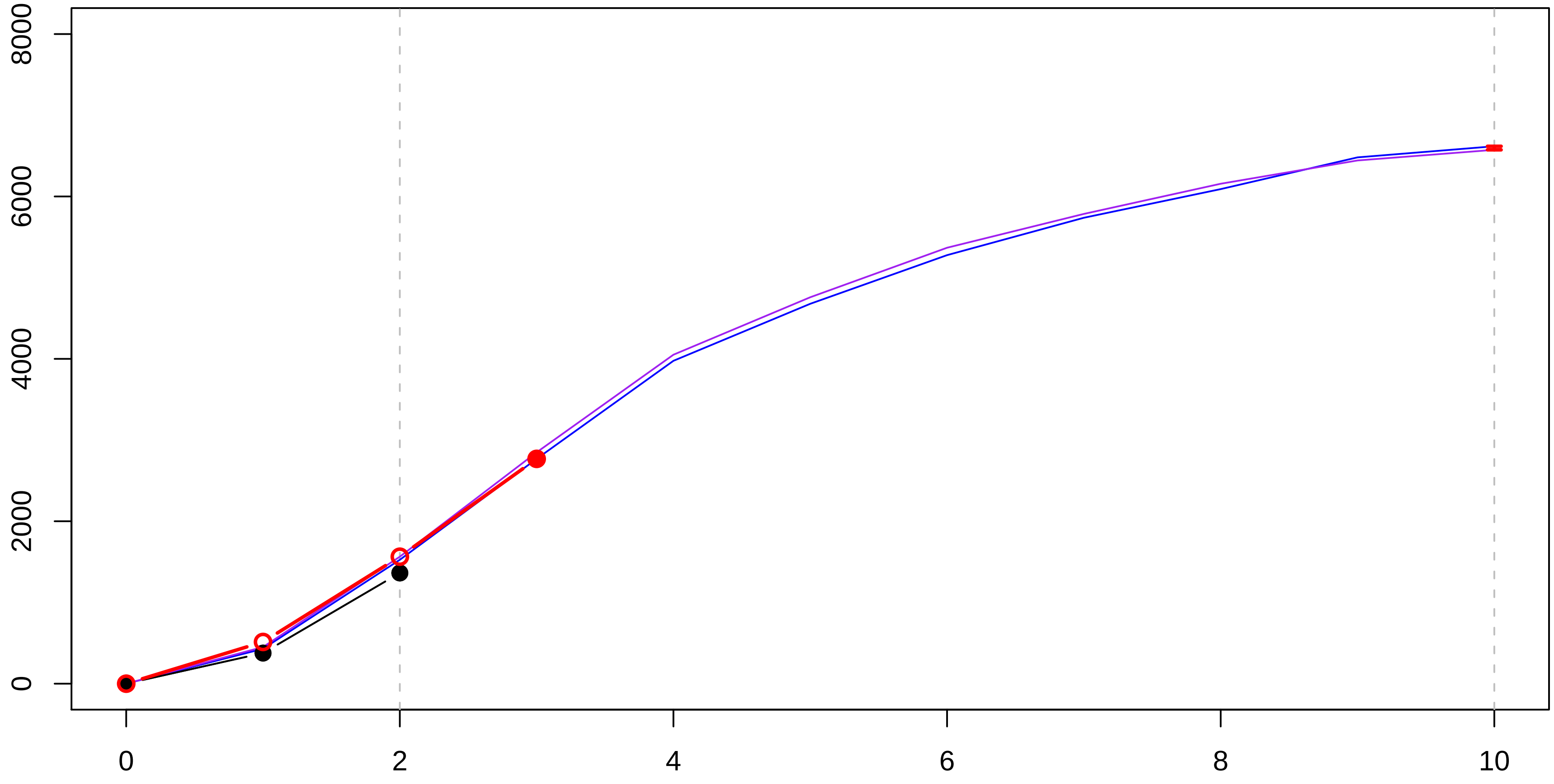
ultimate $(R - \mathbb{E}(R))$ versus one year uncertainty,



Bootstrap and GLM log-Poisson in triangles



Bootstrap and GLM log-Poisson in triangles



Why a Poisson model for IBNR ?

Hachemeister & Stanard (1975), Kremer (1985) and Mack(1991) proved that with a log-Poisson regression model on incremental payments, the sum of predicted payments corresponds to the Chain Ladder estimator.

Recall that $Y_{i,j} \sim \mathcal{P}(L_i + C_j)$, i.e.

- we consider two factors, line L_i and column C_j
- we assume that $\mathbb{E}(Y_{i,j}|\mathcal{F}) = \exp[L_i + C_j]$ (since the link function is log)
- we assume further that $\text{Var}(Y_{i,j}|\mathcal{F}) = \exp[L_i + C_j] = \mathbb{E}(Y_{i,j}$ (since we consider a Poisson regression)

Why a Poisson model for IBNR ?

Adding additional factors is complex (too many parameters, and need to forecast a calendar factor, if any).

Changing the link function is not usual, and having a multiplicative model yield to natural interpretations,

Why not changing the distribution (i.e. the variance function) ?

$\Longrightarrow$ consider Tweedie models.

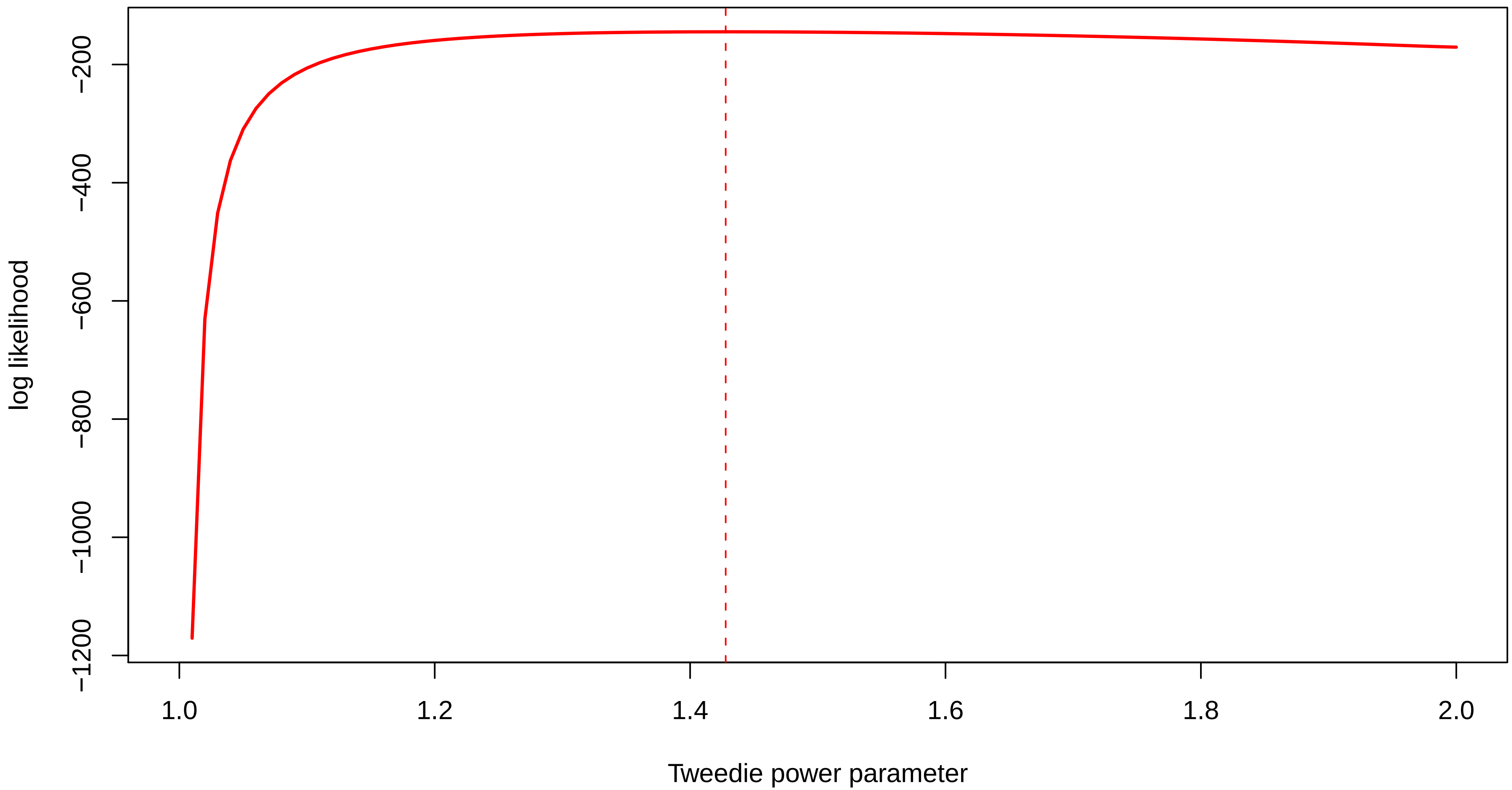
Tweedie models

Assume here that the variance function is $Var(Y) = \varphi\mathbb{E}(Y)^p$ for some $p \in [0,1]$.

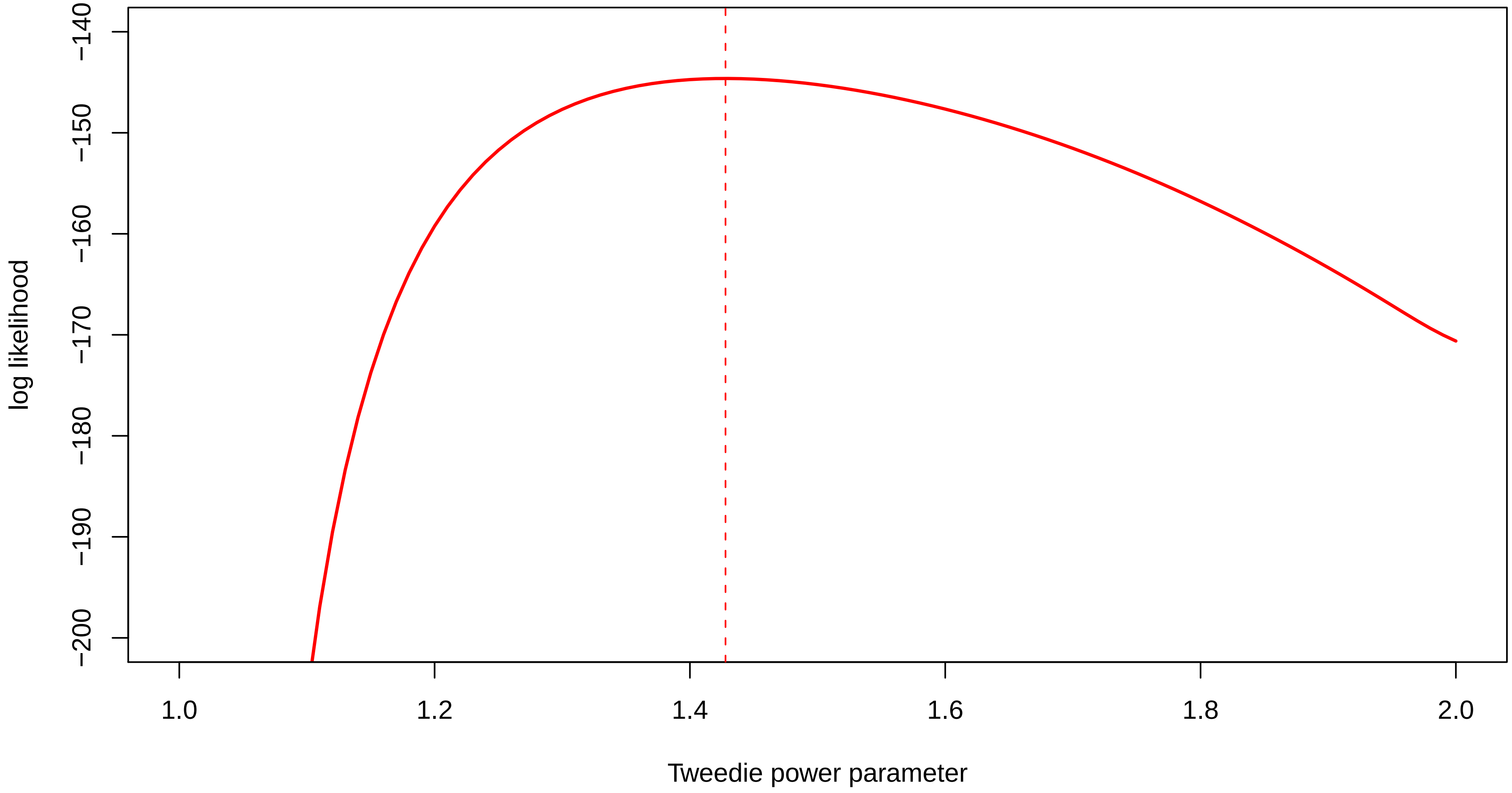
$p = 1$ is obtained with a Poisson model, $p = 2$ with a Gamma model.

If $p \in (1,2)$, we obtain a compound Poisson distribution.

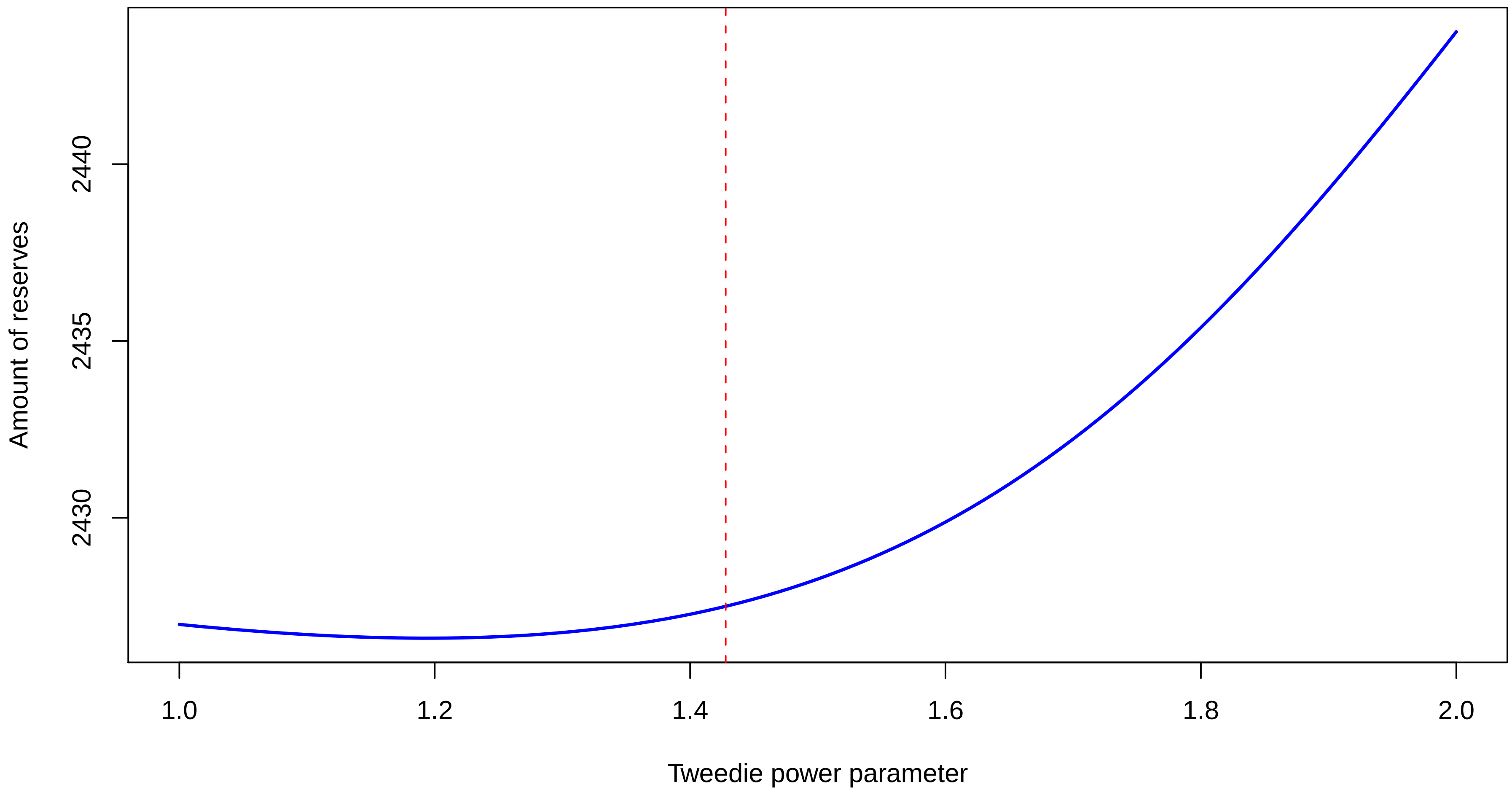
Best estimate and Tweedie parameter



Best estimate and Tweedie parameter



Best estimate and Tweedie parameter



Best estimate and Tweedie parameter

Best estimate amount of reserve with Tweedie power p, with the 95% quantile and the 99.5% quantile