Unifying standard multivariate copulas families (with tail dependence properties)

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inspired by some joint work (and discussion) with

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Agenda

- **Standard copula families**
  - Elliptical distributions (and copulas)
  - Archimedean copulas
  - Extreme value distributions (and copulas)
- **Tail dependence**
  - Tail indexes
  - Limiting distributions
  - Other properties of tail behavior

Copulas

Definition 1
A copula in dimension d is a c.d.f on $[0,1]^d$, with margins $\mathcal{U}([0,1])$.

Theorem 1 1. If C is a copula, and $F_1, ..., F_d$ are univariate c.d.f., then

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_d(x_d)) \ \forall(x_1, ..., x_d) \in \mathbb{R}^d \tag{1}$$

is a multivariate c.d.f. with $F \in \mathcal{F}(F_1, ..., F_d)$.

2. Conversely, if $F \in \mathcal{F}(F_1, ..., F_d)$, there exists a copula C Satisfying (1). This copula is usually not unique, but it is if $F_1, ..., F_d$ are absolutely continuous, and then,

$$C(u_1, ..., u_d) = F(F_1^{-1}(u_1), ..., F_d^{-1}(u_d)), \ \forall(u_1, , ..., u_d) \in [0,1]^d \tag{2}$$

where quantile functions $F_1^{-1}, ..., F_n^{-1}$ are generalized inverse (left cont.) of F_i's.

If $\boldsymbol{X} \sim F$, then $\boldsymbol{U} = (F_1(X_1), \cdots, F_d(X_d)) \sim C$.

Benchmark copulas

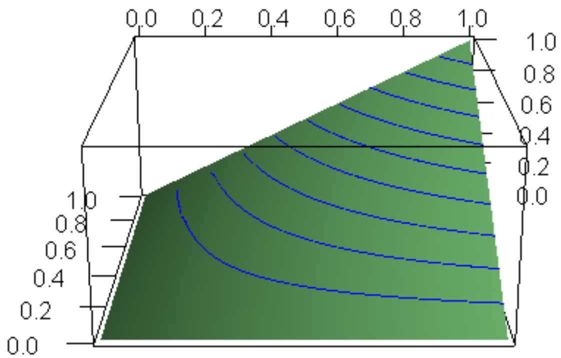
Definition 2

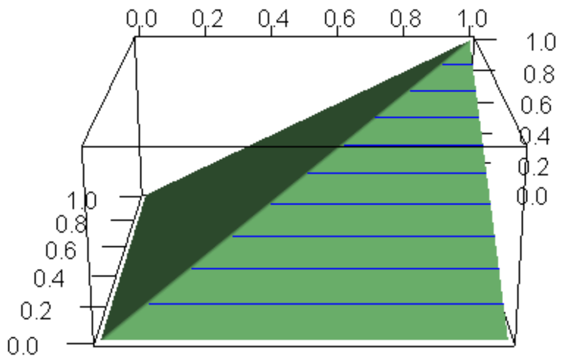
The independent copula $C^{\perp}$ is defined as

$$C^{\perp}(u_1,...,u_n)=u_1\times\cdots\times u_d=\prod_{i=1}^{d}u_i.$$

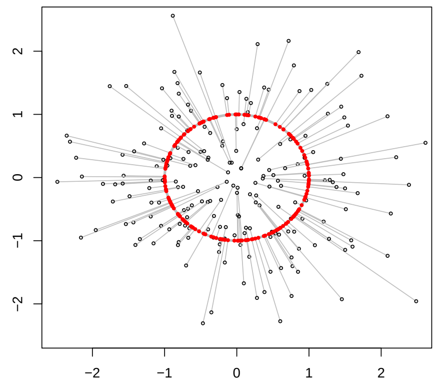
Definition 3

The comonotonic copula C^{+} (the Fréchet-Hoeffding upper bound of the set of copulas) is the copuladefined as $C^{+}(u_1,...,u_d)=\min\{u_1,...,u_d\}$.





Spherical distributions

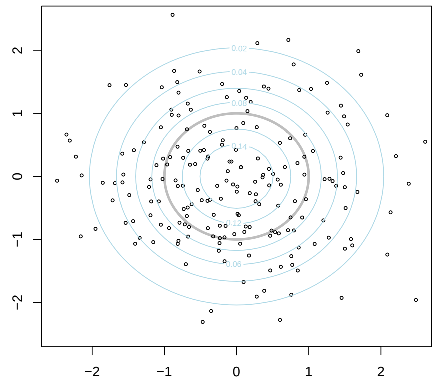


Definition 4

Random vector $\boldsymbol{X}$ as a spherical distribution if

$$\boldsymbol{X} = R \cdot \boldsymbol{U}$$

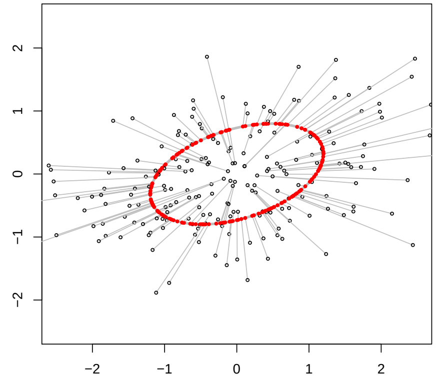
where R is a positive random variable and $\boldsymbol{U}$ is uniformly distributed on the unit sphere of $\mathbb{R}^d$.



E.g. $\boldsymbol{X} \sim \mathcal{N}(\mathbf{0}, \mathbb{I})$.

Those distribution can be non-symmetric, see Hartman & Wintner (AJM, 1940) or Cambanis, Huang & Simons (JMVA, 1979))

Elliptical distributions

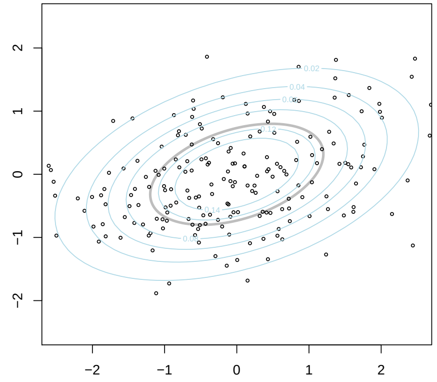


Definition 5

Random vector $\boldsymbol{X}$ as a elliptical distribution if

$$\boldsymbol{X} = \boldsymbol{\mu} + R \cdot \boldsymbol{A} \cdot \boldsymbol{U}$$

where $\boldsymbol{A}$ satisfies $\boldsymbol{A}\boldsymbol{A}' = \boldsymbol{\Sigma}$.



E.g. $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Elliptical distribution are popular in finance, see e.g. Jondeau, Poon & Rockinger (FMPM, 2008)

Archimedean copula

Definition 6

If $d \geq 2$, an Archimedean generator is a function $\phi : [0,1] \to [0,\infty)$ such that ϕ^{-1} is d-completely monotone (i.e. ψ is d-completely monotone if ψ is continuous and $\forall k = 0, 1, ..., d$, $(-1)^k d^k \psi(t)/dt^k \geq 0$).

Definition 7

Copula C is an Archimedean copula is, for some generator ϕ,

$$C(u_1, ..., u_d) = \phi^{-1}(\phi(u_1) + ... + \phi(u_d)), \forall u_1, ..., u_d \in [0,1].$$

Function $h(t) = \exp[-\phi(t)]$ (i.e. $h^{-1}(t) = \phi(-\log[t])$ is called a multiplicative generator of copula C,

$$C(u_1, ..., u_d) = h^{-1}(h(u_1) \times \cdots \times h(u_d)), \forall u_1, ..., u_d \in [0,1].$$

Stochastic representation of Archimedean copulas

see Nešlehová & McNeil (AS, 2009).

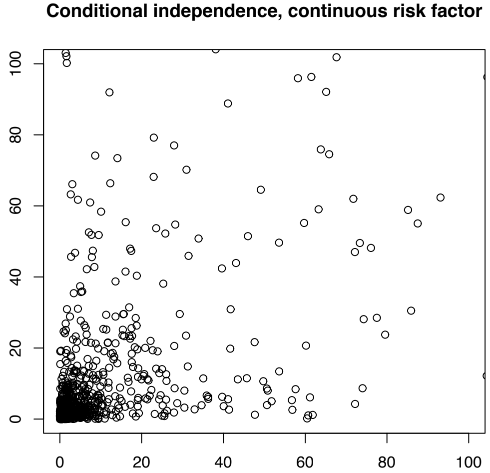
Archimedean copula, exchangeability and frailties

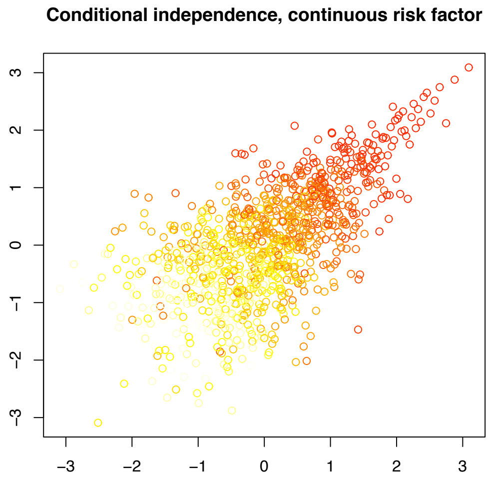
Consider residual life times $\boldsymbol{X} = (X_1, \cdots, X_d)$ conditionally independent given some latent factor Θ, and such that $\mathbb{P}(X_i > x_i | \Theta) = \overline{B}_i(x_i)^\theta$. Then

$$\overline{F}(\boldsymbol{x}) = \mathbb{P}(\boldsymbol{X} > \boldsymbol{x}) = \psi\left(-\sum_{i=1}^{n} \log \overline{F}_i(x_i)\right)$$

where ψ is the Laplace transform of Θ, $\psi(t) = \mathbb{E}(e^{-t\Theta})$. Thus, the survival copula of $\boldsymbol{X}$ is Archimedean, with generator $= \psi^{-1}$.

See Oakes (JASA, 1989).



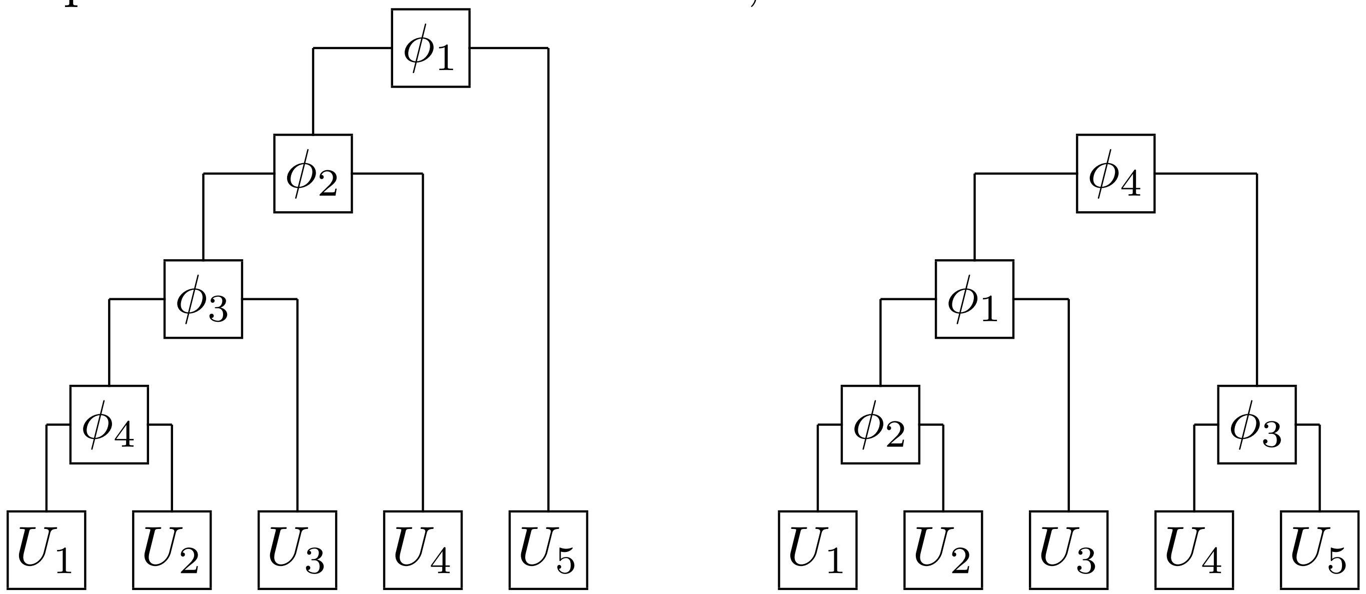


Nested Archimedean copula, and hierarchical structures

Consider $C(u_1, \cdots, u_d)$ defined as

$$\phi_1^{-1}[\phi_1[\phi_2^{-1}(\phi_2[\cdots\phi_{d-1}^{-1}[\phi_{d-1}(u_1)+\phi_{d-1}(u_2)]+\cdots+\phi_2(u_{d-1}))]+\phi_1(u_d)]$$

where ϕ_i's are generators. Then C is a copula if $\phi_i \circ \phi_{i-1}^{-1}$ is the inverse of a Laplace transform, and is called fully nested Archimedean copula. Note that partial nested copulas can also be considered,



(Univariate) extreme value distributions

Central limit theorem, $X_i \sim F$ i.i.d. $\dfrac{\overline{X}_n - b_n}{a_n} \xrightarrow{\mathcal{L}} S$ as $n \to \infty$ where S is a non-degenerate random variable.

Fisher-Tippett theorem, $X_i \sim F$ i.i.d., $\dfrac{X_{n:n} - b_n}{a_n} \xrightarrow{\mathcal{L}} M$ as $n \to \infty$ where M is a non-degenerate random variable.

Then

$$\mathbb{P}\left(\frac{X_{n:n} - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \to G(x) \text{ as } n \to \infty, \forall x \in \mathbb{R}$$

i.e. F belongs to the max domain of attraction of G, G being an extreme value distribution : the limiting distribution of the normalized maxima.

$$-\log G(x) = (1 + \xi x)_+^{-1/\xi}$$

(Multivariate) extreme value distributions

Assume that $\boldsymbol{X}_i \sim F$ i.i.d.,

$$F^n(\boldsymbol{a}_n\boldsymbol{x}+\boldsymbol{b}_n) \to G(\boldsymbol{x}) \text{ as } n\to\infty, \forall \boldsymbol{x}\in\mathbb{R}^d$$

i.e. F belongs to the max domain of attraction of G, G being an (multivariate) extreme value distribution : the limiting distribution of the normalized componentwise maxima,

$$\boldsymbol{X}_{n:n} = (\max\{X_{1,i}\},\cdots,\max\{X_{d,i}\})$$

$$-\log G(\boldsymbol{x}) = \mu([\boldsymbol{0},\boldsymbol{\infty})\backslash[\boldsymbol{0},\boldsymbol{x}]), \forall \boldsymbol{x}\in\mathbb{R}_+^d$$

where μ is the exponent measure. It is more common to use the stable tail dependence function ℓ defined as

$$\ell(\boldsymbol{x}) = \mu([\boldsymbol{0},\boldsymbol{\infty})\backslash[\boldsymbol{0},\boldsymbol{x}^{-1}]), \forall \boldsymbol{x}\in\mathbb{R}_+^d$$

i.e.

$$-\log G(\boldsymbol{x}) = \ell(-\log G_1(x_1), \cdots, \log G_d(x_d)) \forall \boldsymbol{x} \in \mathbb{R}^d$$

Note that there exists a finite measure H on the simplex of $\mathbb{R}^d$ such that

$$\ell(x_1, \cdots, x_d) = \int_{\mathcal{S}_d} \max\{\omega_1 x_1, \cdots, \omega_d x_d\} dH(\omega_1, \cdots, \omega_d)$$

for all $(x_1, \cdots, x_d) \in \mathbb{R}_+^d$, and $\int_{\mathcal{S}_d} \omega_i dH(\omega_1, \cdots, \omega_d) = 1$ for all $i = 1, \cdots, n$.

Definition 8

Copula $C : [0,1]^d \to [0,1]$ is an multivariate extreme value copula if and only if there exists a stable tail dependence function such that ℓ

$$C(u_1, \cdots, u_d) = \exp[-\ell(-\log u_1, \cdots, -\log u_d)]$$

Assume that $\boldsymbol{U}_i \sim C$ i.i.d.,

$$C^n(\boldsymbol{u}^{\frac{1}{n}}) = C^n(u_1^{\frac{1}{n}}, \cdots, u_d^{\frac{1}{n}}) \to \Gamma(\boldsymbol{u}) \text{ as } n \to \infty, \forall \boldsymbol{x} \in \mathbb{R}^d$$

i.e. C belongs to the max domain of attraction of Γ, Γ being an (multivariate) extreme value copula.

What do we have in dimension 2 ?

C is an Archimedean copula if $C(u,v)=\psi(\psi^{-1}(u)+\psi^{-1}(v))$

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Quantifying tail dependence, in dimension 2 ?

Venter (2002) suggested to visualize tail concentration functions,

Definition 9

For the lower tail, define

$$L(z) = \frac{\mathbb{P}(U < z, V < z)}{z} = \frac{C(z,z)}{z} = \mathbb{P}(U < z | V < z) = \mathbb{P}(V < z | U < z),$$

and for the upper tail

$$R(z) = \frac{\mathbb{P}(U > z, V > z)}{1 - z} = \mathbb{P}(U > z | V > z).$$

Joe (1999) defined tail dependence coefficients from lower and upper limits, respectively (if those limits exist)

$$\lambda_U = R(1) = \lim_{z \to 1} R(z) \text{ et } \lambda_L = L(0) = \lim_{z \to 0} L(z).$$

Quantifying tail dependence, in dimension 2 ?

Definition 10

Let (X,Y) denote a random vector in $\mathbb{R}^2$. Define tail dependence indices in the lower (L) and upper (U) tails as

$$\lambda_L = \lim_{u\downarrow 0} \mathbb{P}\left(X \leq F_X^{-1}(u) \,|Y \leq F_Y^{-1}(u)\right) \in [0,1],$$

and

$$\lambda_U = \lim_{u\uparrow 1} \mathbb{P}\left(X > F_X^{-1}(u) \,|Y > F_Y^{-1}(u)\right) \in [0,1].$$

Proposition 1

Let (X,Y) denote a random vector with copula C, then

$$\lambda_L = \lim_{u\downarrow 0} \frac{C(u,u)}{u} \text{ and } \lambda_U = \lim_{u\downarrow 0} \frac{C^\star(u,u)}{u}.$$

Quantifying tail dependence, in dimension 2 ?

Exemple1

For Archimedean copulas (see Nelsen (2007), C. & Segers (JMVA, 2008)),

$$\lambda_U = 2 - \lim_{x \to 0} \frac{1 - \phi^{-1}(2x)}{1 - \phi^{-1}(x)} \text{ and } \lambda_L = \lim_{x \downarrow 0} \frac{\phi^{-1}(2\phi(x))}{x} = \lim_{x \downarrow \infty} \frac{\phi^{-1}(2x)}{\phi^{-1}(x)}.$$

Ledford and Tawn (B, 1996) suggested an alternative approach : assume that $X \overset{\mathcal{L}}{=} Y$.

- assuming independence, $\mathbb{P}(X > t, Y > t) = \mathbb{P}(X > t) \times \mathbb{P}(Y > t) = \mathbb{P}(X > t)^2$,
- assuming comonotonicity, $\mathbb{P}(X > t, Y > t) = \mathbb{P}(X > t) = \mathbb{P}(X > t)^1$,

Thus, assume that one has $\mathbb{P}(X > t, Y > t) \sim \mathbb{P}(X > t)^\eta$ as $t \to \infty$, where $\eta \in [1, 2]$ will be a tail dependence index.

Quantifying tail dependence, in dimension 2 ?

Following Coles, Heffernan & Tawn (E, 1999) define

Definition 11

Let

$$\overline{\chi}_U(z) = \frac{2\log(1-z)}{\log C^\star(z,z)} - 1 \text{ et } \overline{\chi}_L(z) = \frac{2\log(1-z)}{\log C(z,z)} - 1$$

Then $\eta_U = (1 + \lim_{z\to 0} \overline{\chi}_U(z))/2$ and $\eta_L = (1 + \lim_{z\to 0} \overline{\chi}_L(z))/2$ are respectively tail indices in the upper and lower tail, respectively.

Exemple2

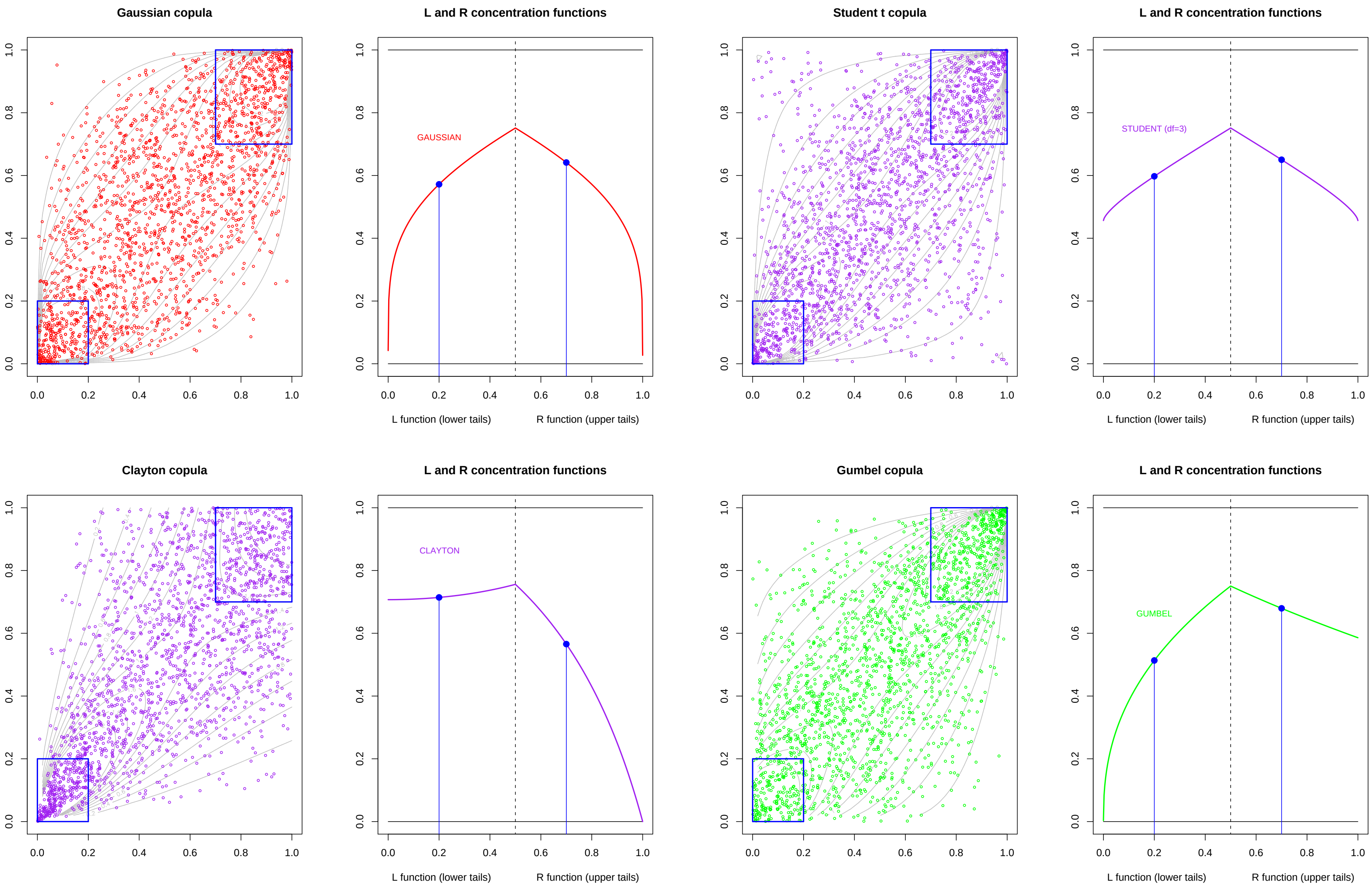
If (X, Y) has a Gumbel copula, with (unit) Fréchet margins

$$\mathbb{P}(X \leq x, Y \leq y) = \exp(-(x^{-\alpha} + y^{-\alpha})^{1/\alpha}), \text{ where } \alpha \geq 0, \forall x, y \geq 0$$

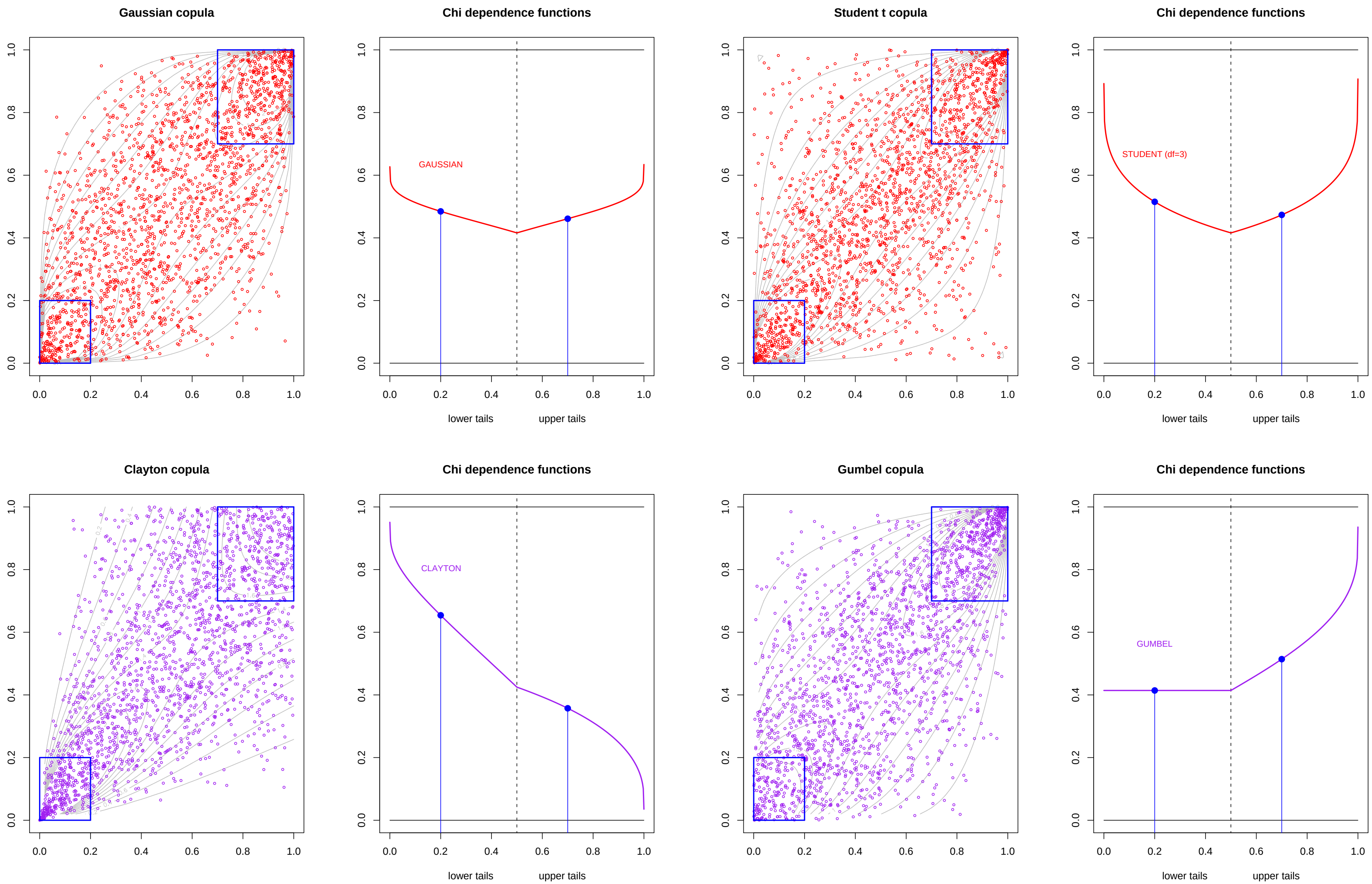
then $\eta_U = 1$ while $\eta_L = 1/2^\alpha$.

For a Gaussian copula with correlation r $\eta_U = \eta_L = (1 + r)/2$.

Quantifying tail dependence, in dimension 2 ?



Quantifying tail dependence, in dimension 2 ?

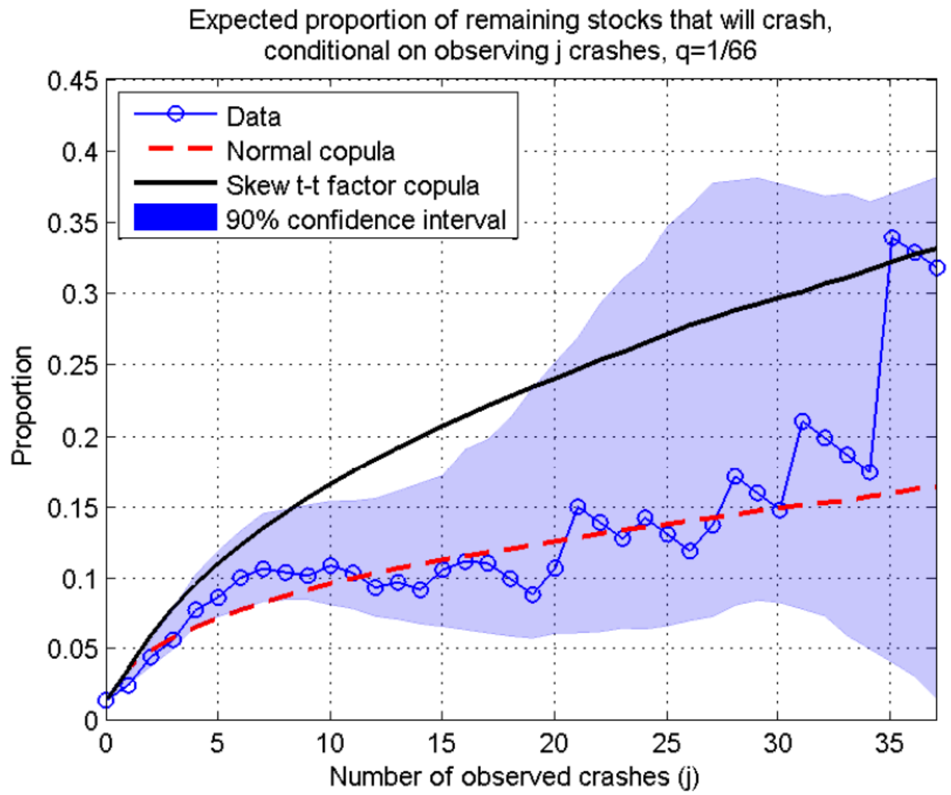


Can describe tail dependence in dimension $d \geq 2$?

Oh & Patton (2012) defined a crash dependence index (related to a measure in Embrechts, et al., 2000) :

let $N_u = \sum_{i=1}^{d} \mathbf{1}(X_i \leq F_i^{-1}(u))$, define

$$\pi_{u,k} = \frac{\mathbb{E}[N_n|N_u \geq k] - k}{d - k}$$



(**Source** : Oh & Patton (2012))