Archimax Copulas

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based on joint work with

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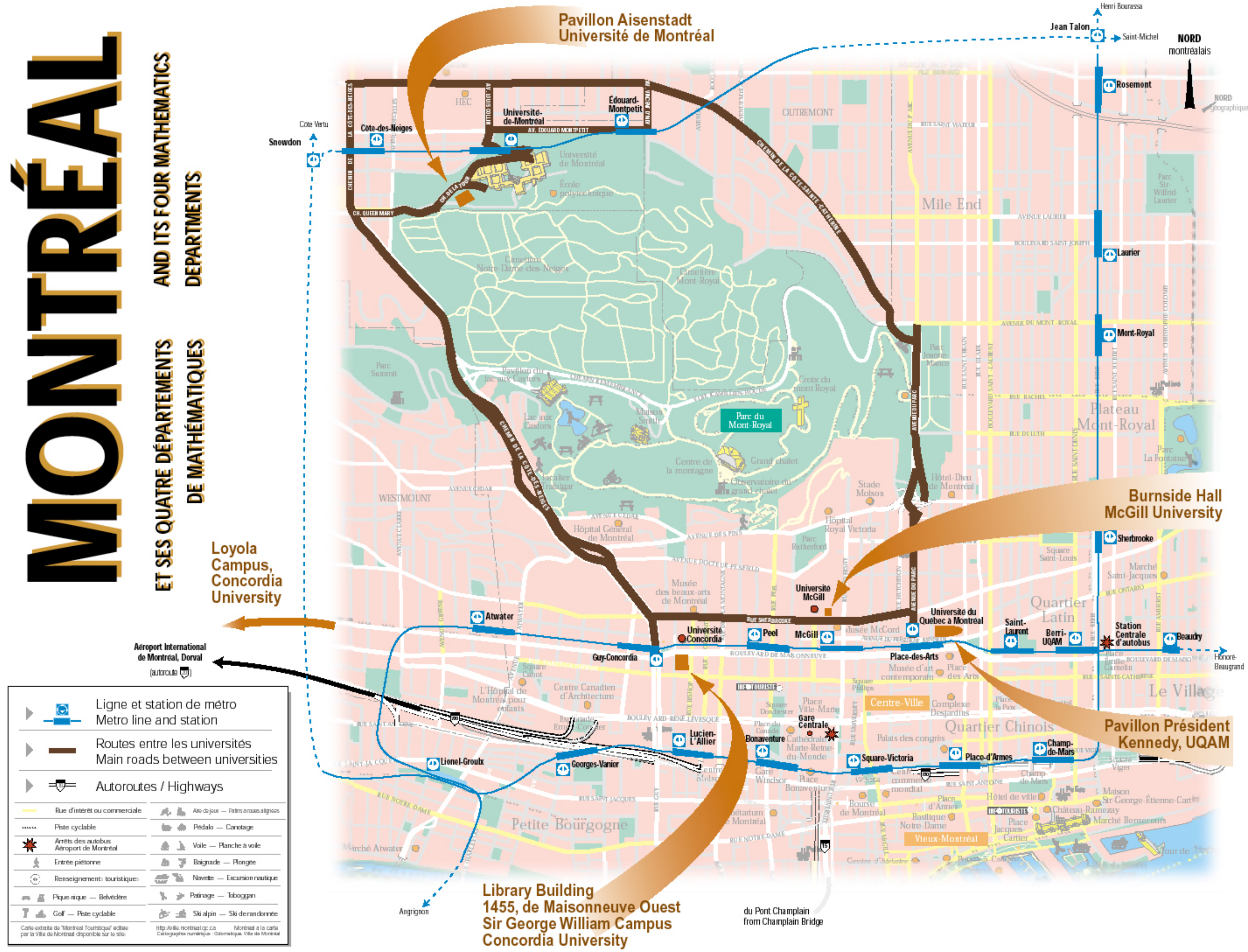
March 2014, CIMAT, Guanajuato, Mexico.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Burnside Hall
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Agenda

- Copulas
- **Standard copula families**
- Elliptical distributions (and copulas)
- Archimedean copulas
- Extreme value distributions (and copulas)
- **Archimax copulas**
- Archimax copulas in dimension 2
- Archimax copulas in dimension $d \geq 3$

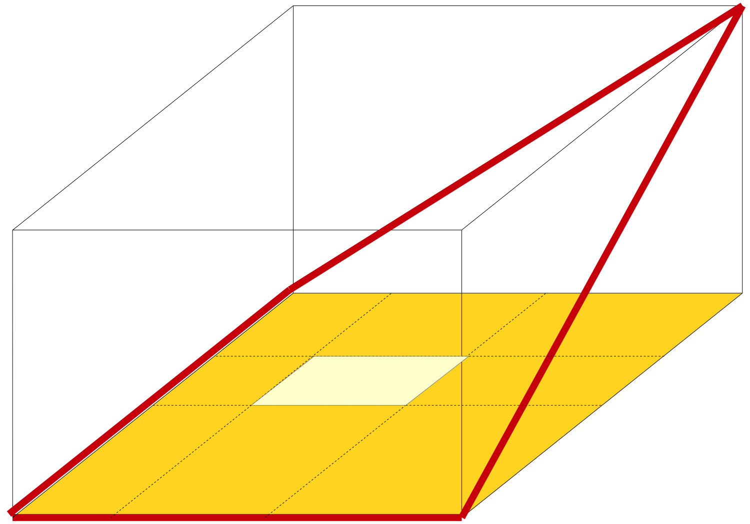
Copulas, in dimension $d = 2$

Definition 1

A copula in dimension 2 is a c.d.f on $[0,1]^2$, with margins $\mathcal{U}([0,1])$.

Thus, let $C(u,v) = \mathbb{P}(U \leq u, V \leq v)$, where $0 \leq u, v \leq 1$, then

- $C(0,x) = C(x,0) = 0 \quad \forall x \in [0,1]$,
- $C(1,x) = C(x,1) = x \quad \forall x \in [0,1]$,
- and some *increasingness* property



Copulas, in dimension $d = 2$

Definition 2

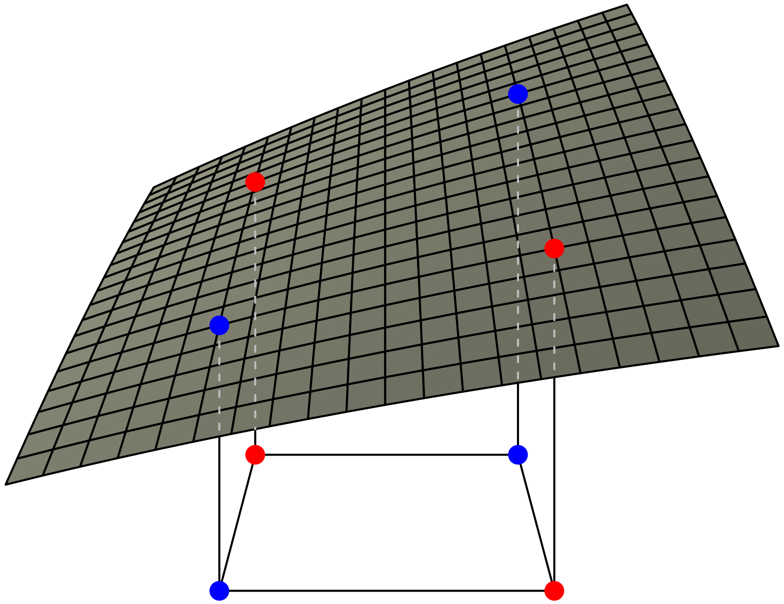
A copula in dimension 2 is a c.d.f on $[0,1]^2$, with margins $\mathcal{U}([0,1])$.

Thus, let $C(u,v) = \mathbb{P}(U \leq u, V \leq v)$, where $0 \leq u, v \leq 1$, then

- $C(0,x) = C(x,0) = 0 \quad \forall x \in [0,1]$,
- $C(1,x) = C(x,1) = x \quad \forall x \in [0,1]$,
- If $0 \leq u_1 \leq u_2 \leq 1$, $0 \leq v_1 \leq v_2 \leq 1$

$$C(u_2,v_2)+C(u_1,v_1) \geq C(u_1,v_2)+C(u_2,v_1)$$

(concept of 2-increasing function in $\mathbb{R}^2$)



see $C(u,v) = \displaystyle\int_0^v \int_0^u \underbrace{c(x,y)}_{\geq 0} \mathrm{d}x\mathrm{d}y$ with the density notation.

Copulas, in dimension $d \geq 2$

The concept of d-increasing function simply means that

$$\mathbb{P}(a_1 \leq U_1 \leq b_1, ..., a_d \leq U_d \leq b_d) = \mathbb{P}(\boldsymbol{U} \in [\boldsymbol{a}, \boldsymbol{b}]) \geq 0$$

where $\boldsymbol{U} = (U_1, ..., U_d) \sim C$ for all $\boldsymbol{a} \leq \boldsymbol{b}$ (where $a_i \leq bi$).

Definition 3

Function $h : \mathbb{R}^d \to \mathbb{R}$ is d-increasing if for all rectangle $[\boldsymbol{a}, \boldsymbol{b}] \subset \mathbb{R}^d$, $V_h([\boldsymbol{a}, \boldsymbol{b}]) \geq 0$, where

$$V_h([\boldsymbol{a}, \boldsymbol{b}]) = \Delta_{\boldsymbol{a}}^{\boldsymbol{b}} h(\boldsymbol{t}) = \Delta_{a_d}^{b_d} \Delta_{a_{d-1}}^{b_{d-1}} ... \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1} h(\boldsymbol{t}) \tag{1}$$

and for all $\boldsymbol{t}$, with

$$\Delta_{a_i}^{b_i} h(\boldsymbol{t}) = h(t_1, ..., t_{i-1}, b_i, t_{i+1}, ..., t_n) - h(t_1, ..., t_{i-1}, a_i, t_{i+1}, ..., t_n). \tag{2}$$

Copulas, in dimension $d \geq 2$

Definition 4

A copula in dimension d is a c.d.f on $[0,1]^d$, with margins $\mathcal{U}([0,1])$.

Theorem 1 1. If C is a copula, and $F_1, ..., F_d$ are univariate c.d.f., then

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_d(x_d)) \;\; \forall (x_1, ..., x_d) \in \mathbb{R}^d \tag{3}$$

is a multivariate c.d.f. with $F \in \mathcal{F}(F_1, ..., F_d)$.

2. Conversely, if $F \in \mathcal{F}(F_1, ..., F_d)$, there exists a copula C satisfying (3). This copula is usually not unique, but it is if $F_1, ..., F_d$ are absolutely continuous, and then,

$$C(u_1, ..., u_d) = F(F_1^{-1}(u_1), ..., F_d^{-1}(u_d)), \;\; \forall (u_1, , ..., u_d) \in [0,1]^d \tag{4}$$

where quantile functions $F_1^{-1}, ..., F_n^{-1}$ are generalized inverse (left cont.) of F_i's.

If $\boldsymbol{X} \sim F$, then $\boldsymbol{U} = (F_1(X_1), \cdots, F_d(X_d)) \sim C$.

Survival (or dual) copulas

Theorem 2 1. If $C^\star$ is a copula, and $\overline{F}_1, ..., \overline{F}_d$ are univariate s.d.f., then

$$\overline{F}(x_1, ..., x_n) = C^\star(\overline{F}_1(x_1), ..., \overline{F}_d(x_d)) \;\; \forall(x_1, ..., x_d) \in \mathbb{R}^d \tag{5}$$

is a multivariate s.d.f. with $F \in \mathcal{F}(F_1, ..., F_d)$.

2. Conversely, if $F \in \mathcal{F}(F_1, ..., F_d)$, there exists a copula $C^\star$ satisfying (5). This copula is usually not unique, but it is if $F_1, ..., F_d$ are absolutely continuous, and then,

$$C^\star(u_1, ..., u_d) = \overline{F}(\overline{F}_1^{-1}(u_1), ..., \overline{F}_d^{-1}(u_d)), \;\; \forall(u_1, , ..., u_d) \in [0, 1]^d \tag{6}$$

where quantile functions $F_1^{-1}, ..., F_n^{-1}$ are generalized inverse (left cont.) of F_i's.

If $\boldsymbol{X} \sim F$, then $\boldsymbol{U} = (F_1(X_1), \cdots, F_d(X_d)) \sim C$ and $\mathbf{1} - \boldsymbol{U} \sim C^\star$.

Benchmark copulas

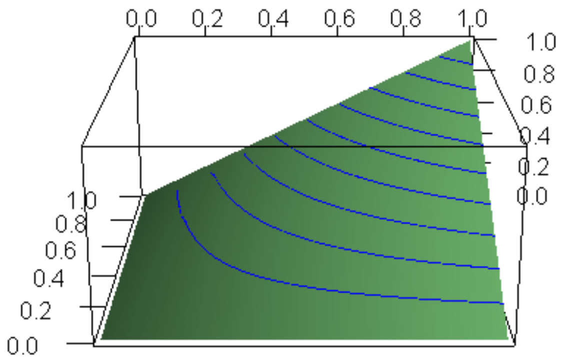
Definition 5

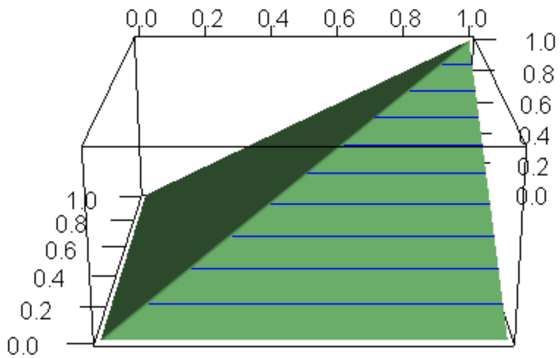
The independent copula $C^{\perp}$ is defined as

$$C^{\perp}(u_1, ..., u_n) = u_1 \times \cdots \times u_d = \prod_{i=1}^{d} u_i.$$

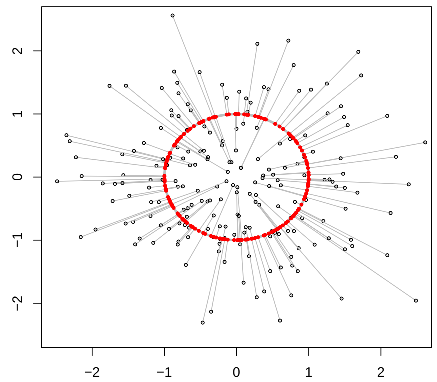
Definition 6

The comonotonic copula C^{+} (the Fréchet-Hoeffding upper bound of the set of copulas) is the copula defined as $C^{+}(u_1, ..., u_d) = \min\{u_1, ..., u_d\}$.





Spherical distributions

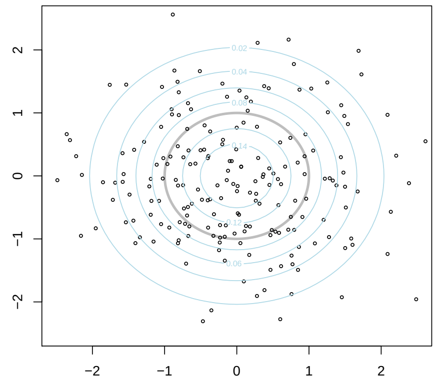


Definition 7

Random vector $\boldsymbol{X}$ as a spherical distribution if

$$\boldsymbol{X} = R \cdot \boldsymbol{U}$$

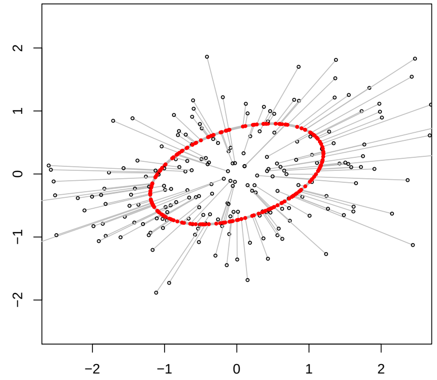
where R is a positive random variable and $\boldsymbol{U}$ is uniformly distributed on the unit sphere of $\mathbb{R}^d$, with $R \perp\!\!\!\perp \boldsymbol{U}$.



E.g. $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{0}, \mathbb{I})$.

Those distribution can be non-symmetric, see Hartman & Wintner (AJM, 1940) or Cambanis, Huang & Simons (JMVA, 1979))

Elliptical distributions

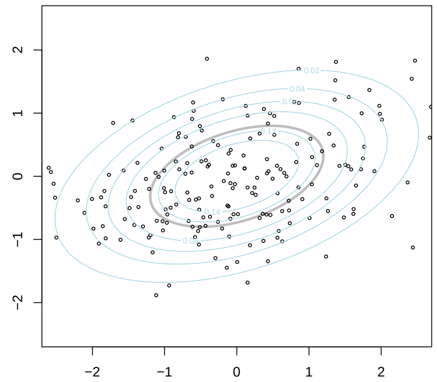


Definition 8

Random vector $\boldsymbol{X}$ as a elliptical distribution if

$$\boldsymbol{X} = \boldsymbol{\mu} + R \cdot \boldsymbol{A} \cdot \boldsymbol{U}$$

where R is a positive random variable and $\boldsymbol{U}$ is uniformly distributed on the unit sphere of $\mathbb{R}^d$, with $R \perp\!\!\!\perp \boldsymbol{U}$, and where $\boldsymbol{A}$ satisfies $\boldsymbol{A}\boldsymbol{A}' = \boldsymbol{\Sigma}$.



E.g. $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Elliptical distribution are popular in finance, see e.g. Jondeau, Poon & Rockinger (FMPM, 2008)

Archimedean copula

Definition 9

If $d \geq 2$, an Archimedean generator is a function $\phi : [0,1] \to [0,\infty)$ such that ϕ^{-1} is d-completely monotone (i.e. ψ is d-completely monotone if ψ is continuous and $\forall k = 0, 1, ..., d$, $(-1)^k \mathrm{d}^k \psi(t)/\mathrm{d}t^k \geq 0$).

Definition 10

Copula C is an Archimedean copula is, for some generator ϕ,

$$C(u_1, ..., u_d) = \phi^{-1}[\phi(u_1) + ... + \phi(u_d)], \forall u_1, ..., u_d \in [0,1].$$

Exemple1

$\phi(t) = -\log(t)$ yields the independent copula $C^{\perp}$.

$\phi(t) = [-\log(t)]^{\theta}$ yields Gumbel copula C_{θ} (note that $\psi(t) = \phi^{-1}(t) = \exp[-t^{1/\theta}]$).

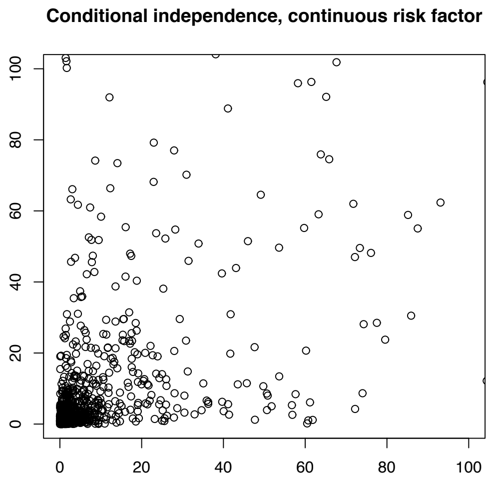
Archimedean copula, exchangeability and frailties

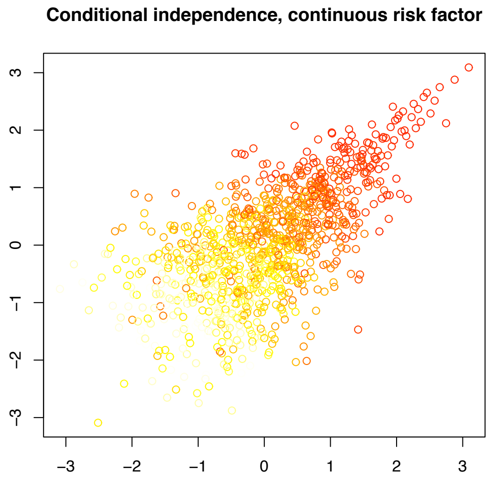
Consider residual life times $\boldsymbol{X} = (X_1, \cdots, X_d)$ conditionally independent given some latent factor Θ, and such that $\mathbb{P}(X_i > x_i|\Theta = \theta) = \overline{B}_i(x_i)^\theta$. Then

$$\overline{F}(\boldsymbol{x}) = \mathbb{P}(\boldsymbol{X} > \boldsymbol{x}) = \psi\left(-\sum_{i=1}^{n}\log \overline{F}_i(x_i)\right)$$

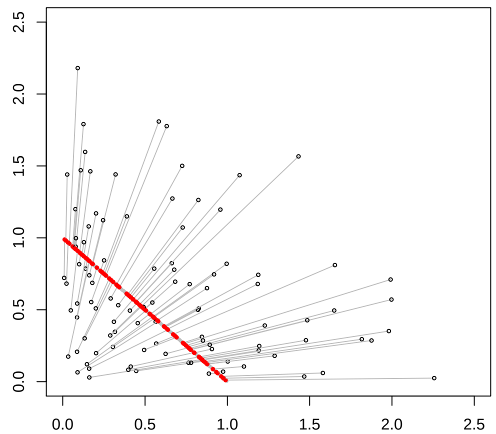
where ψ is the Laplace transform of Θ, $\psi(t) = \mathbb{E}(e^{-t\Theta})$. Thus, the survival copula of $\boldsymbol{X}$ is Archimedean, with generator $\phi = \psi^{-1}$.

See Oakes (JASA, 1989).



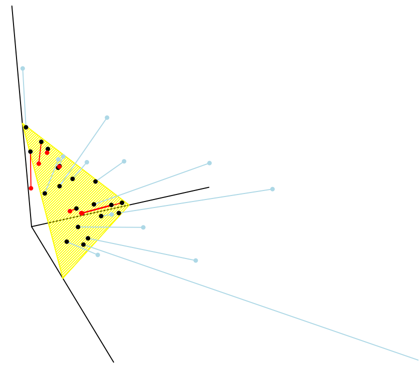


Stochastic representation of Archimedean copulas



Consider some striclty positive random variable R independent of $\boldsymbol{U}$, uniform on the simplex of $\mathbb{R}^d$. The survival copula of $\boldsymbol{X} = R\cdot\boldsymbol{U}$ is Archimedean, and its generator is the inverse of Williamson d-transform,

$$\phi^{-1}(t) = \int_x^\infty \left(1 - \frac{x}{t}\right)^{d-1} \mathrm{d}F_R(t).$$



Note that $R \overset{\mathcal{L}}{=} \phi(U_1) + \cdots + \phi(U_d)$.

See Nešlehová & McNeil (AS, 2009).

Archimedean copula and distortion

Definition 11

Function $h : [0,1] \to [0,1]$ defined as $h(t) = \exp[-\phi(t)]$ is called a distortion function.

Genest & Rivest (SPL, 2001), Morillas (M, 2005) considered distorted copulas (also called *multivariate probability integral transformation*)

Definition 12

Let h be some distortion function, and C a copula, then

$$C_h(u_1, ..., u_d) = h^{-1}(C(h(u_1), \cdots, h(u_d)))$$

is a copula.

Exemple2

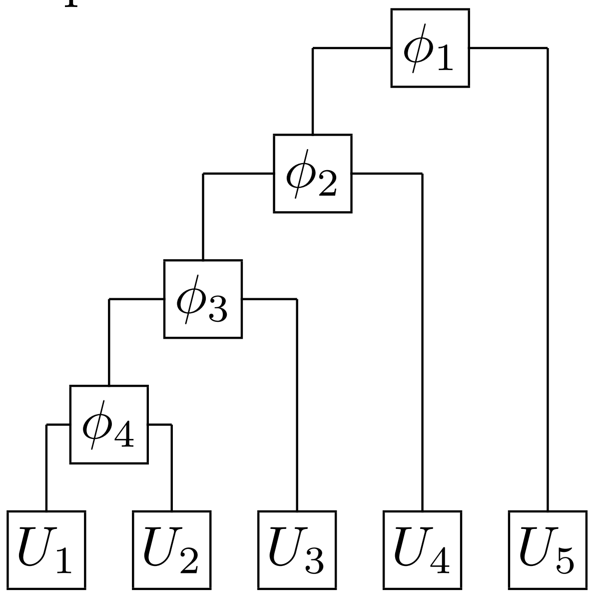
If $C = C^{\perp}$, then $C_h^{\perp}$ is the Archimedean copula with generator $\phi(t) = -\log h(t)$.

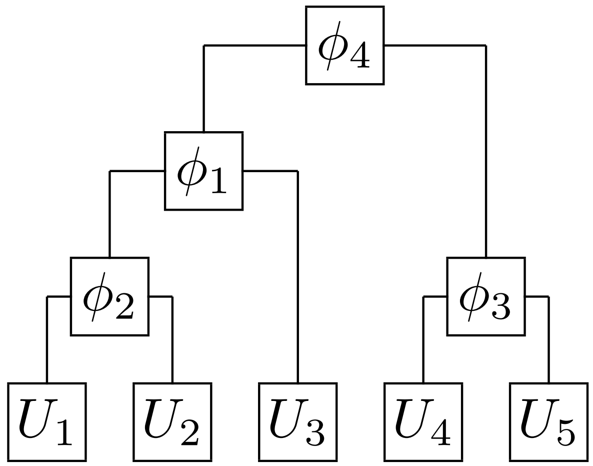
Nested Archimedean copula, and hierarchical structures

Consider $C(u_1, \cdots, u_d)$ defined as

$$\phi_1^{-1}[\phi_1[\phi_2^{-1}(\phi_2[\cdots\phi_{d-1}^{-1}[\phi_{d-1}(u_1)+\phi_{d-1}(u_2)]+\cdots+\phi_2(u_{d-1}))]+\phi_1(u_d)]$$

where ϕ_i's are generators. Then C is a copula if $\phi_i \circ \phi_{i-1}^{-1}$ is the inverse of a Laplace transform, and is called fully nested Archimedean copula. Note that partial nested copulas can also be considered,





(Univariate) extreme value distributions

Central limit theorem, $X_i \sim F$ i.i.d. $\dfrac{\overline{X}_n - b_n}{a_n} \xrightarrow{\mathcal{L}} S$ as $n \to \infty$ where S is a non-degenerate random variable.

Fisher-Tippett theorem, $X_i \sim F$ i.i.d., $\dfrac{X_{n:n} - b_n}{a_n} \xrightarrow{\mathcal{L}} M$ as $n \to \infty$ where M is a non-degenerate random variable.

Then

$$\mathbb{P}\left(\frac{X_{n:n} - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \to G(x) \text{ as } n \to \infty, \forall x \in \mathbb{R}$$

i.e. F belongs to the max domain of attraction of G, G being an extreme value distribution : the limiting distribution of the normalized maxima.

$$-\log G(x) = (1 + \xi x)_+^{-1/\xi}$$

(Multivariate) extreme value distributions

Assume that $\boldsymbol{X}_i \sim F$ i.i.d.,

$$F^n(\boldsymbol{a}_n \boldsymbol{x} + \boldsymbol{b}_n) \to G(\boldsymbol{x}) \text{ as } n \to \infty, \forall \boldsymbol{x} \in \mathbb{R}^d$$

i.e. F belongs to the max domain of attraction of G, G being an (multivariate) extreme value distribution : the limiting distribution of the normalized componentwise maxima,

$$\boldsymbol{X}_{n:n} = (\max\{X_{1,i}\}, \cdots, \max\{X_{d,i}\})$$

$$-\log G(\boldsymbol{x}) = \mu([\boldsymbol{0}, \boldsymbol{\infty})\backslash[\boldsymbol{0}, \boldsymbol{x}]), \forall \boldsymbol{x} \in \mathbb{R}_+^d$$

where μ is the exponent measure. It is more common to use the stable tail dependence function ℓ defined as

$$\ell(\boldsymbol{x}) = \mu([\boldsymbol{0}, \boldsymbol{\infty})\backslash[\boldsymbol{0}, \boldsymbol{x}^{-1}]), \forall \boldsymbol{x} \in \mathbb{R}_+^d$$

i.e.

$$-\log G(\boldsymbol{x}) = \ell(-\log G_1(x_1), \cdots, \log G_d(x_d)),\ \forall \boldsymbol{x} \in \mathbb{R}^d$$

Note that there exists a finite measure H on the simplex of $\mathbb{R}^d$ such that

$$\ell(x_1, \cdots, x_d) = \int_{\mathcal{S}_d} \max\{\omega_1 x_1, \cdots, \omega_d x_d\} dH(\omega_1, \cdots, \omega_d)$$

for all $(x_1, \cdots, x_d) \in \mathbb{R}_+^d$, and $\int_{\mathcal{S}_d} \omega_i dH(\omega_1, \cdots, \omega_d) = 1$ for all $i = 1, \cdots, n$.

Definition 13

Copula $C : [0,1]^d \to [0,1]$ is an multivariate extreme value copula if and only if there exists a stable tail dependence function such that ℓ

$$C_\ell(u_1, \cdots, u_d) = \exp[-\ell(-\log u_1, \cdots, -\log u_d)]$$

Assume that $\boldsymbol{U}_i \sim \Gamma$ i.i.d.,

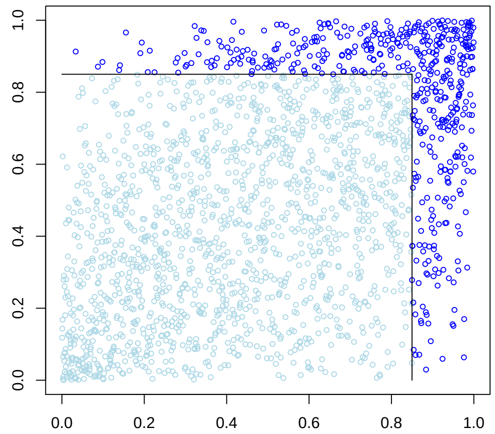
$$\Gamma^n(\boldsymbol{u}^{\frac{1}{n}}) = \Gamma^n(u_1^{\frac{1}{n}}, \cdots, u_d^{\frac{1}{n}}) \to C_\ell(\boldsymbol{u}) \text{ as } n \to \infty, \forall \boldsymbol{x} \in \mathbb{R}^d$$

i.e. Γ belongs to the max domain of attraction of C_ℓ, C_ℓ being an (multivariate) extreme value copula, $\Gamma \in \text{MDA}(C_\ell)$.

The stable tail dependence function $\ell(\cdot)$

Observe that

$$n\left[1-C\left(1-\frac{x_1}{n},\cdots,1-\frac{x_d}{n}\right)\right]\rightarrow\underbrace{-\log\left[\Gamma(e^{-x_1},\cdots,e^{-x_1})\right]}_{=\ell(\boldsymbol{x})}$$



Exemple3

Gumbel copula, $\theta\in[1,+\infty]$,

$$\ell_\theta(x_1,\cdots,x_d)=\left(x_1^\theta+\cdots+x_d^\theta\right)^{1/\theta}=\|\boldsymbol{x}\|_\theta\ \forall\boldsymbol{x}\in\mathbb{R}_+^d$$

Function $\ell(\cdot)$ statisfies

$$\underbrace{\max\{x_1,\cdots,x_d\}}_{\substack{\text{(asympt.) comonotonicity}\\ \ell_\infty(\boldsymbol{x})}}\leq\ell(\boldsymbol{x})\leq\underbrace{x_1+\cdots+x_d}_{\substack{\text{(asympt.) independence}\\ \ell_1(\boldsymbol{x})}}$$

The stable tail dependence function $\ell(\cdot)$

Function $\ell(\cdot)$ is homogeneous, $\ell(t \cdot \boldsymbol{x}) = t \cdot \ell(\boldsymbol{x})$ $\forall t \in \mathbb{R}_+$.

$\longrightarrow$ consider the restriction of $\ell(\cdot)$ on the unit simplex Δ_{d-1},

$$\ell(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 \cdot \underbrace{\ell\left(\frac{x_1}{\|\boldsymbol{x}\|}, \cdots, \frac{x_d}{\|\boldsymbol{x}\|}\right)}_{\ell(\boldsymbol{\omega})} = \|\boldsymbol{x}\|_1 \cdot A(\omega_1, \cdots, \omega_{d-1})$$

where $A(\cdot)$ is Pickands dependence function. Observe that

$$\max\{\omega_1, \cdots, \omega_{d-1}, \omega_d\} \leq A(\omega_1, \cdots, \omega_{d-1}) \leq 1, \quad \forall \boldsymbol{\omega} \in \Delta_{d-1}$$

What do we have in dimension 2 ?

C is an Archimedean copula if $C = C_\phi$

$$C_\phi(u,v) = \phi^{-1}\left[\phi(u)+\phi(v)\right]$$

C is an extreme value copula if $C = C_A = C_\ell$

$$\begin{cases} C_A(u,v) = \exp\left(\log[uv]A\left(\dfrac{\log[v]}{\log[uv]}\right)\right) \\ C_\ell(u,v) = \exp[-\ell(-\log u, -\log v)] \end{cases}$$

where $A : [0,1] \to [1/2,1]$ is Pickands dependence function, convex, with

$$\max\{\omega, 1-\omega\} \leq A(\omega) \leq 1, \forall \omega \in [0,1].$$

Exemple4

$A(\omega) = 1$ yields the independent copula, $C^\perp$.

What do we have in dimension 2 ?

Exemple5

$\phi(t) = [-\log(t)]^\theta$ yields Gumbel copula C_θ.

$A(\omega) = \left[\omega^\theta + (1-\omega)^\theta\right]^{1/\theta}$ yields Gumbel copula C_θ.

Definition 14

C is an Archimax copula (from Capéerà, Fougères & Genest (JMVA, 2000)) if $C = C_{\phi,A}$

$$C_{\phi,A}(u,v) = \phi^{-1}\left[[\phi(u)+\phi(v)]A\left(\frac{\phi(u)}{\phi(u)+\phi(v)}\right)\right]$$

Note that there is a frailty type construction, see C. (K, 2006) : given Θ, $\boldsymbol{X}$ has (survival) copula C_A, Θ has Laplace transform ϕ^{-1}.

Note that $C_{\phi,A}$ is the distorted version of copula C_A.

What do we have in dimension $d \geq 3$?

Definition 15

C is an Archimax copula (from C., Fougères, Genest & Nešlehová (JMVA, 2014)) if $C = C_{\phi,\ell}$

$$C_{\phi,\ell}(u_1, \cdots, u_d) = \phi^{-1}\left[\ell(\phi(u_1) + \cdots + \phi(u_d))\right]$$

This function *is* a copula function.

Stochastic representation of Archimax copulas

Theorem 3

$C_{\phi,\ell}$ is the survival copula of $\boldsymbol{X} = \boldsymbol{T}/\Theta$ where Θ has Laplace transform ϕ^{-1}, independent of random vector $\boldsymbol{T}$ satisfying

$$\mathbb{P}(\boldsymbol{T} > \boldsymbol{t}) = \exp[-\ell(\boldsymbol{t})] = C_\ell(e^{-\boldsymbol{t}}).$$

(see also Li (JMVA, 2009) and Marshall & Olkin (JASA, 1988)).

Limiting behavior of Archimax copulas

One can wonder what would be the max-domain of attraction of that copula ?

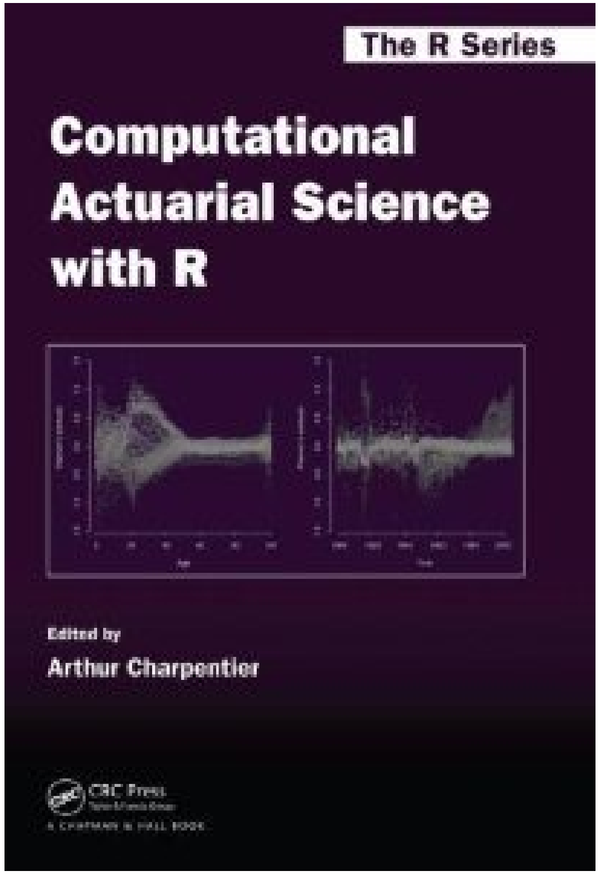
$$C_{\phi,\ell} \in \text{MDA}(C_{\ell^\star})$$

If $\psi = \phi^{-1}$ is such that $\psi(1-s)$ is regularly varying at 0 with index $\theta \in [1, +\infty]$, then $C_{\phi,\ell}$ belongs to the max domain of attraction of

$$C_{\ell^\star}(u_1, \cdots, u_d) = \exp\left[-\ell^{\frac{1}{\theta}}\left(|\log(u_1)|^\theta, \cdots, |\log(u_d)|^\theta\right)\right]$$

(see also C. & Segers (JMVA, 2009) and Larsson & Nešlehová (AAP, 2011) in the case of Archimedean copulas).

forthcoming book (April 2014),
Computational Actuarial Science with R



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