Extremes and dependence in the context of Solvency II for insurance companies

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On risk dependence in QIS's

The BSCR is determined as follows:

$$BSCR = \sqrt{\sum_{rxc} CorrSCR_{r,c} \bullet SCR_r \bullet SCR_c} - min(\sqrt{\sum_{rxc} CorrSCR_{r,c} \bullet KC_r \bullet KC_c}, FDB)$$

where

$CorrSCR_{r,c}$ = the cells of the correlation matrix CorrSCR

SCR_r, SCR_c = capital charges for the individual SCR risks according to the rows and columns of the correlation matrix CorrSCR

KC_r, KC_c = risk mitigation effects for the individual SCR risks[15]

and CorrSCR is defined as follows:

CorrSCR=	SCR_{mkt}	SCR_{def}	SCR_{life}	SCR_{health}	SCR_{nl}
SCR_{mkt}	1				
SCR_{def}	0.25	1			
SCR_{life}	0.25	0.25	1		
SCR_{health}	0.25	0.25	0.25	1	
SCR_{nl}	0.25	0.5	0	0	1

http ://www.ceiops.eu/media/files/consultations/QIS/QIS3/QIS3TechnicalSpecificationsPart1.PDF

On risk dependence in QIS's

The market sub-risks should be combined to an overall charge SCR_{mkt} for market risk using a correlation matrix as follows:

$$SCR_{mkt} = \sqrt{\sum_{rxc} CorrMkt_{r,c} \bullet Mkt_r \bullet Mkt_c}$$

where

$CorrMkt_{r,c}$ = the cells of the correlation matrix CorrMkt

Mkt_r, Mkt_c = capital charges for the individual market risks according to the rows and columns of the correlation matrix CorrMkt

and the correlation matrix *CorrMkt* is defined as:[18]

CorrMkt	Mkt_{int}	Mkt_{eq}	Mkt_{prop}	Mkt_{sp}	Mkt_{conc}	Mkt_{fx}
Mkt_{int}	1					
Mkt_{eq}	0	1				
Mkt_{prop}	0.5	0.75	1			
Mkt_{sp}	0.25	0.25	0.25	1		
Mkt_{conc}	0	0	0	0	1	
Mkt_{fx}	0.25	0.25	0.25	0.25	0	1

http ://www.ceiops.eu/media/files/consultations/QIS/QIS3/QIS3TechnicalSpecificationsPart1.PDF

On risk dependence in QIS's

CorrLife=	$Life_{mort}$	$Life_{long}$	$Life_{dis}$	$Life_{lapse}$	$Life_{exp}$	$Life_{rev}$	$Life_{CAT}$
$Life_{mort}$	1						
$Life_{long}$	0	1					
$Life_{dis}$	0.5	0	1				
$Life_{lapse}$	0	0.25	0	1			
$Life_{exp}$	0.25	0.25	0.5	0.5	1		
$Life_{rev}$	0	0.25	0	0	0.25	1	
$Life_{CAT}$	0	0	0	0	0	0	1

CorrMCR=	MCR_{mkt}	MCR_{life}	MCR_{nl}	MCR_{health}
MCR_{mkt}	1			
MCR_{life}	0.25	1		
MCR_{nl}	0.25	0	1	
MCR_{health}	0.25	0.25	0	1

http ://www.ceiops.eu/media/files/consultations/QIS/QIS3/QIS3TechnicalSpecificationsPart1.PDF

On risk dependence in QIS's

The correlation matrix CorrLob_{pr} is specified as follows:

$CorrLob_{pr}$=	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1: A (workers' comp)	1														
2: A (health)	0,5	1													
3:A (other)H	0,5	0,5	1												
4: M (3rd party)	0,25	0,25	0,25	1											
5: M (other)	0,25	0,25	0,25	0,5	1										
6: MAT	0,25	0,25	0,25	0,5	0,25	1									
7: Fire	0,25	0,25	0,25	0,25	0,25	0,25	1								
8: 3rd party liab	0,5	0,25	0,25	0,5	0,25	0,25	0,25	1							
9: credit	0,25	0,25	0,25	0,25	0,25	0,25	0,25	0,5	1						
10: legal exp.	0,5	0,25	0,5	0,5	0,5	0,25	0,25	0,5	0,5	1					
11: assistance	0,25	0,25	0,25	0,25	0,5	0,5	0,5	0,25	0,25	0,25	1				
12: misc.	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	1			
13: reins. (prop)	0,25	0,25	0,25	0,25	0,25	0,25	0,5	0,25	0,25	0,25	0,5	0,25	1		
14: reins. (cas)	0,25	0,25	0,25	0,25	0,25	0,25	0,25	0,5	0,5	0,5	0,25	0,25	0,25	1	
15: reins. (MAT)	0,25	0,25	0,25	0,25	0,25	0,5	0,5	0,25	0,25	0,25	0,25	0,5	0,25	0,25	1

http ://www.ceiops.eu/media/files/consultations/QIS/QIS3/QIS3TechnicalSpecificationsPart1.PDF

How to capture dependence in risk models ?

Is correlation relevant to capture dependence information ?

Consider (see McNeil, Embrechts & Straumann (2003)) 2 log-normal risks,

- $X \sim LN(0,1)$, i.e. $X = \exp(X^\star)$ where $X^\star \sim \mathcal{N}(0,1)$
- $Y \sim LN(0,\sigma^2)$, i.e. $Y = \exp(Y^\star)$ where $Y^\star \sim \mathcal{N}(0,\sigma^2)$

Recall that $corr(X^\star, Y^\star)$ takes *any* value in $[-1,+1]$.

Since $corr(X,Y) \neq corr(X^\star, Y^\star)$, what can be $corr(X,Y)$?

How to capture dependence in risk models ?

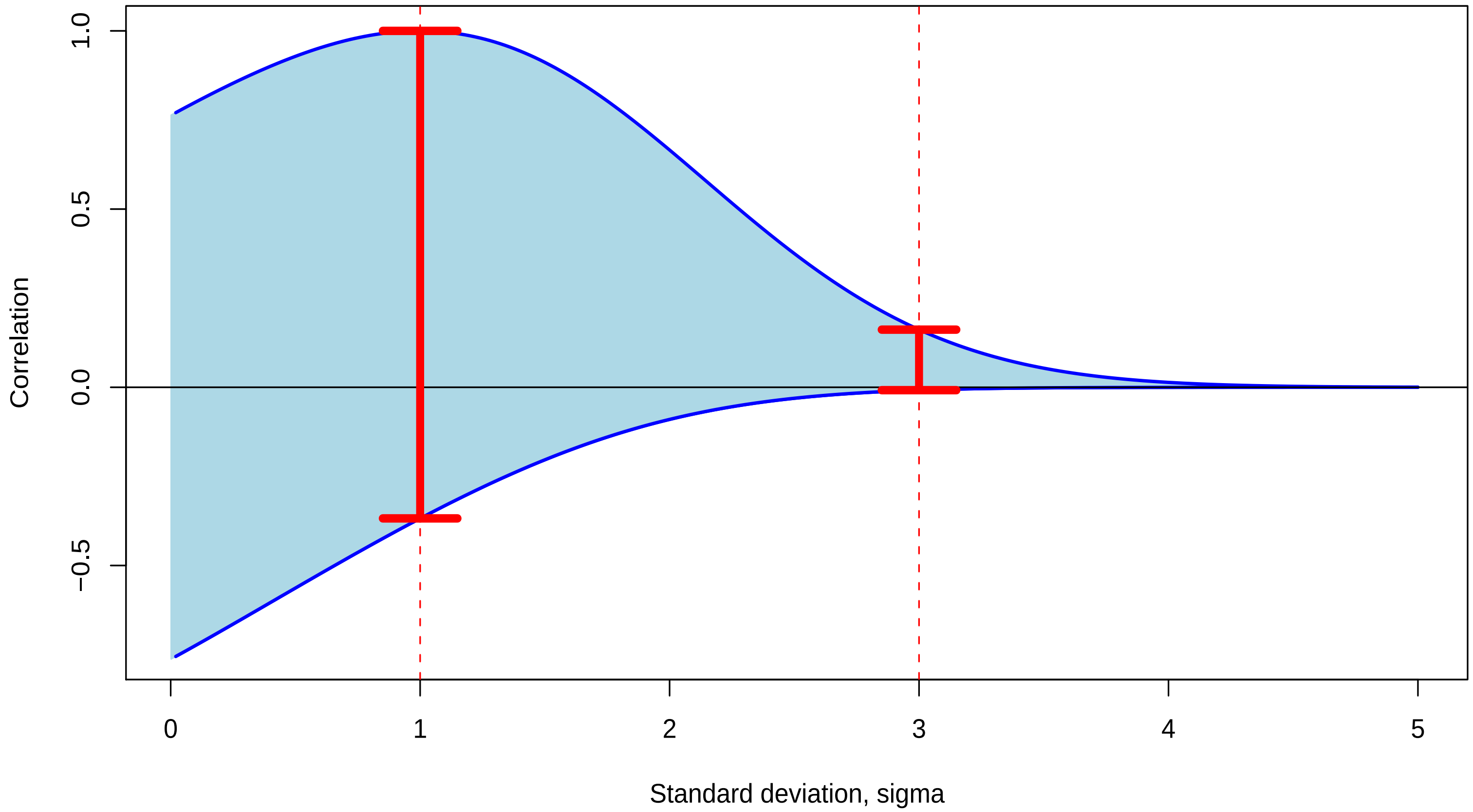

Fig. 1 – Range for the correlation, $cor(X,Y)$, $X \sim LN(0,1)$,$Y \sim LN(0,\sigma^2)$.

How to capture dependence in risk models ?

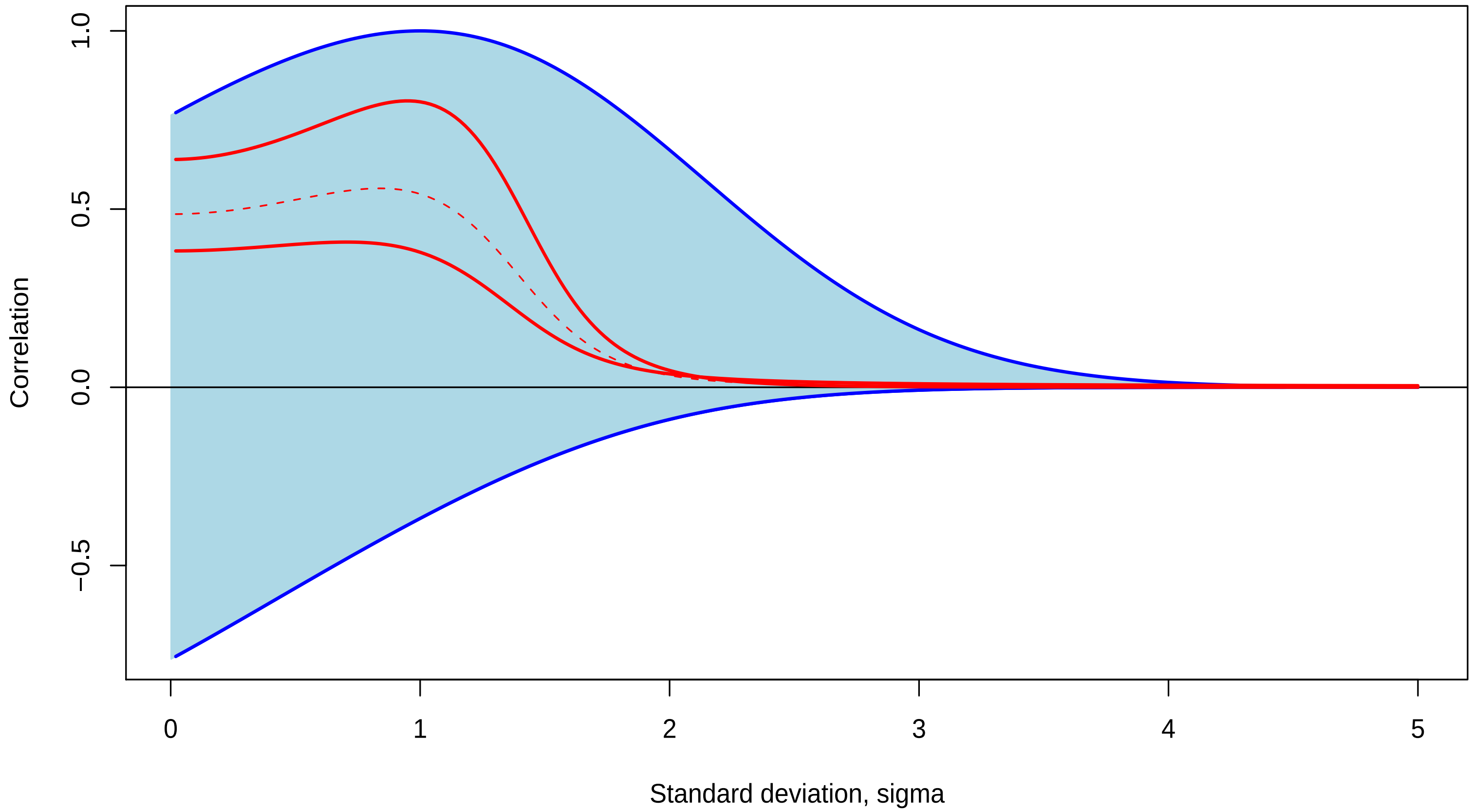

Fig. 2 – $cor(X,Y)$, $X \sim LN(0,1)$,$Y \sim LN(0,\sigma^2)$, Gaussian copula, $r = 0.5$.

What about official actuarial documents ?

ASSOCIATION ACTUARIELLE INTERNATIONALE

INTERNATIONAL ACTUARIAL ASSOCIATION

MEASUREMENT OF LIABILITIES FOR INSURANCE CONTRACTS:

CURRENT ESTIMATE AND RISK MARGINS

24 March 2008

MEASUREMENT OF LIABILITIES FOR INSURANCE CONTRACTS:
CURRENT ESTIMATES AND RISK MARGINS – MARCH 2008 RE-EXPOSURE DRAFT
IAA ad hoc Risk Margin Working Group

The diversification factors are based on the experience of a AA rated entity with, on average, a positive risk profile. Diversification effects at a group level are allocated on a marginal basis. The results for the risks are given in Table B.6.

For the quantile method, it is assumed that the underlying risks are independent (i.e., no tail correlation adjustment is needed at the 75% level).

What about official actuarial documents ?

MEASUREMENT OF LIABILITIES FOR INSURANCE CONTRACTS:
CURRENT ESTIMATES AND RISK MARGINS – MARCH 2008 RE-EXPOSURE DRAFT
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APPENDIX C – Diversification

C2 Technical approaches

In the *Blue Book*, the IAA proposes the use of copulas as the theoretically correct method to calculate diversification effects. Indeed in general we can say that the use of a "standard" correlation matrix is wrong. Copula functions have the advantage that they can be used to accurately combine other distributions than from the "normal family". They also recognize dependencies that change in the tail of the distributions.

What about official actuarial documents ?

Severe incidents can impact risks that are normally independent. An example: normally market risk and mortality risk will be independent. But in case of a severe pandemic like the Spanish flu would happen with millions of deaths worldwide this will certainly have economic consequences and will also impact market risk, for example equity risk. In practice, combining several distributions implies that the dependency in the tail is greater than average risks. In applying copula functions this can be handled, while in a standard correlation matrix this is not possible.

However, copula functions are rather complex to use, particularly in case a large number of distributions have to be combined. A practical solution can be to adjust the correlation matrix in such a way that the confidence level we are interested in, the combined distribution results are reasonably correct. The adjusted correlation factors are also called “tail-correlations”. More background of this simplified approach can be found in Group Consultatif (2005).

What about regulatory technical documents ?

QIS3

Calibration of the underwriting risk, market risk and MCR

CEIOPS

Committee of European Insurance and Occupational Pensions Supervisors

CEIOPS- FS-14/07

What about regulatory technical documents ?

1.84 In view of the insufficiency of currently available data, the setting of these correlation coefficients will necessarily include a certain degree of judgement. This is also true because, when selecting correlation coefficients, allowance should be made for non-linear tail correlation, which is not captured under a "pure" linear correlation approach.[9] To allow for this, the correlations used should be higher than simple analysis of relevant data would indicate.

9 For example, two risk variables X and Y may have zero linear correlation, but may nonetheless be dependent "in the tail" (i.e. in the occurrence of adverse events). In fact, such a situation is not uncommon for variables related to insurance risk. In such cases, the correlation matrix used in the standard formula to aggregate the risk capital charges for the two risks should be set to capture such tail dependence, i.e. the related correlation coefficient should be set higher than zero. Note that a similar assumption was made in QIS2 with respect to the dependence between premium and reserve risk.

What about regulatory technical documents ?

1.84 the setting of these correlation coefficients will necessarily include a certain degree of judgement. allowance should be made for non-linear tail correlation, which is not captured under a "pure" linear correlation approach.[9] To allow for this, the correlations used should be higher than simple analysis of relevant data would indicate.

9 For example, two risk variables X and Y may have zero linear correlation, but may nonetheless be dependent "in the tail" (i.e. in the occurrence of adverse events).

What about regulatory technical documents ?

1.84

non-linear tail correlation,

the correlations used should be higher than simple analysis of relevant data would indicate.

Motivations : dependence and copulas

Definition 1. *A copula C is a joint distribution function on $[0,1]^d$, with uniform margins on $[0,1]$.*

Theorem 2. ***(Sklar)*** *Let C be a copula, and $F_1,\ldots,F_d$ be d marginal distributions, then $F(\boldsymbol{x}) = C(F_1(x_1),\ldots,F_d(x_d))$ is a distribution function, with $F \in \mathcal{F}(F_1,\ldots,F_d)$.*

Conversely, if $F \in \mathcal{F}(F_1,\ldots,F_d)$, there exists C such that $F(\boldsymbol{x}) = C(F_1(x_1),\ldots,F_d(x_d))$. Further, if the F_i's are continuous, then C is unique, and given by

$$C(\boldsymbol{u}) = F(F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)) \text{ for all } u_i \in [0,1]$$

We will then define the copula of F, or the copula of $\boldsymbol{X}$.

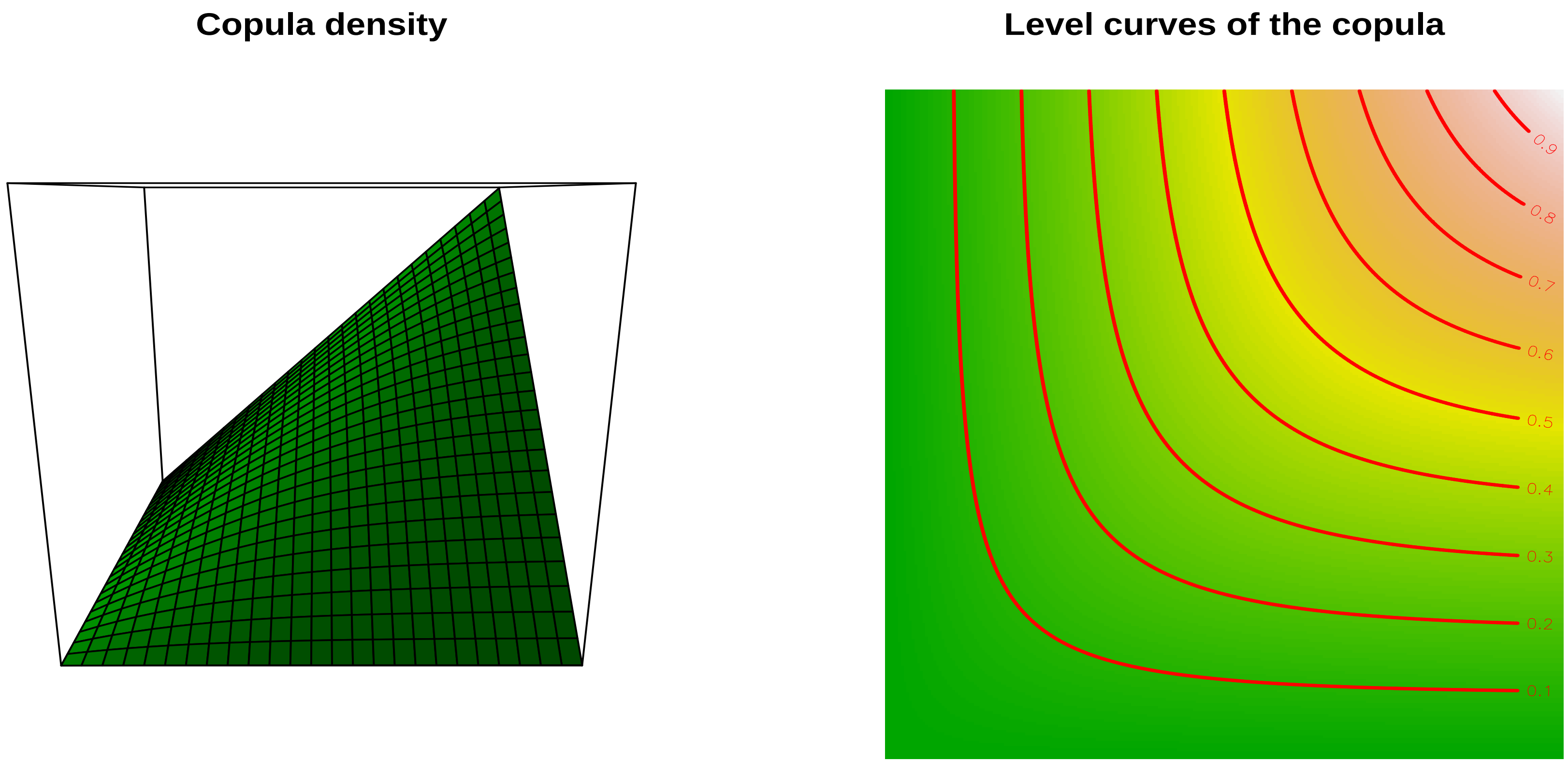

Fig. 3 – Graphical representation of a copula, $C(u,v) = \mathbb{P}(U \leq u, V \leq v)$.

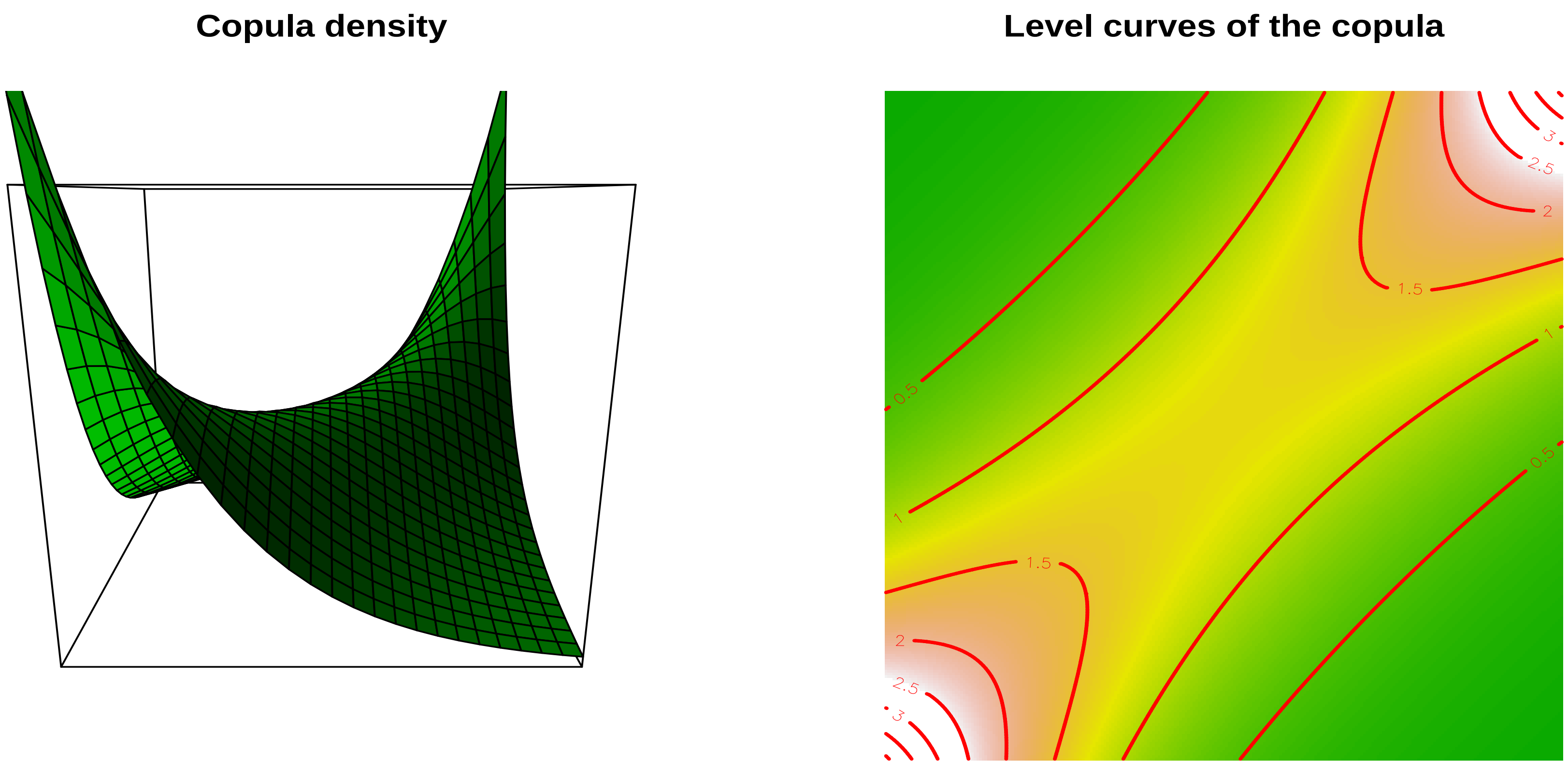

Fig. 4 – Density of a copula, $c(u,v) = \dfrac{\partial^2 C(u,v)}{\partial u \partial v}$.

Some very classical copulas

- The independent copula $C(u, v) = uv = C^{\perp}(u, v)$.

The copula is standardly denoted Π, P or $C^{\perp}$, and an independent version of (X, Y) will be denoted $(X^{\perp}, Y^{\perp})$. It is a random vector such that $X^{\perp} \overset{\mathcal{L}}{=} X$ and $Y^{\perp} \overset{\mathcal{L}}{=} Y$, with copula $C^{\perp}$.

In higher dimension, $C^{\perp}(u_1, \ldots, u_d) = u_1 \times \ldots \times u_d$ is the independent copula.

- The comonotonic copula $C(u, v) = \min\{u, v\} = C^{+}(u, v)$.

The copula is standardly denoted M, or C^{+}, and an comonotone version of (X, Y) will be denoted (X^{+}, Y^{+}). It is a random vector such that $X^{+} \overset{\mathcal{L}}{=} X$ and $Y^{+} \overset{\mathcal{L}}{=} Y$, with copula C^{+}.

(X, Y) has copula C^{+} if and only if there exists a strictly increasing function h such that $Y = h(X)$, or equivalently $(X, Y) \overset{\mathcal{L}}{=} (F_X^{-1}(U), F_Y^{-1}(U))$ where U is $\mathcal{U}([0, 1])$.

Some very classical copulas

In higher dimension, $C^+(u_1, \ldots, u_d) = \min\{u_1, \ldots, u_d\}$ is the comonotonic copula.

- The contercomotonic copula $C(u, v) = \max\{u + v - 1, 0\} = C^-(u, v)$.

The copula is standardly denoted W, or C^-, and an contercomontone version of (X, Y) will be denoted (X^-, Y^-). It is a random vector such that $X^- \overset{\mathcal{L}}{=} X$ and $Y^- \overset{\mathcal{L}}{=} Y$, with copula C^-.

(X, Y) has copula C^- if and only if there exists a strictly decreasing function h such that $Y = h(X)$, or equivalently $(X, Y) \overset{\mathcal{L}}{=} (F_X^{-1}(1 - U), F_Y^{-1}(U))$.

In higher dimension, $C^-(u_1, \ldots, u_d) = \max\{u_1 + \ldots + u_d - (d - 1), 0\}$ is not a copula.

But note that for any copula C,

$$C^-(u_1, \ldots, u_d) \leq C(u_1, \ldots, u_d) \leq C^+(u_1, \ldots, u_d)$$

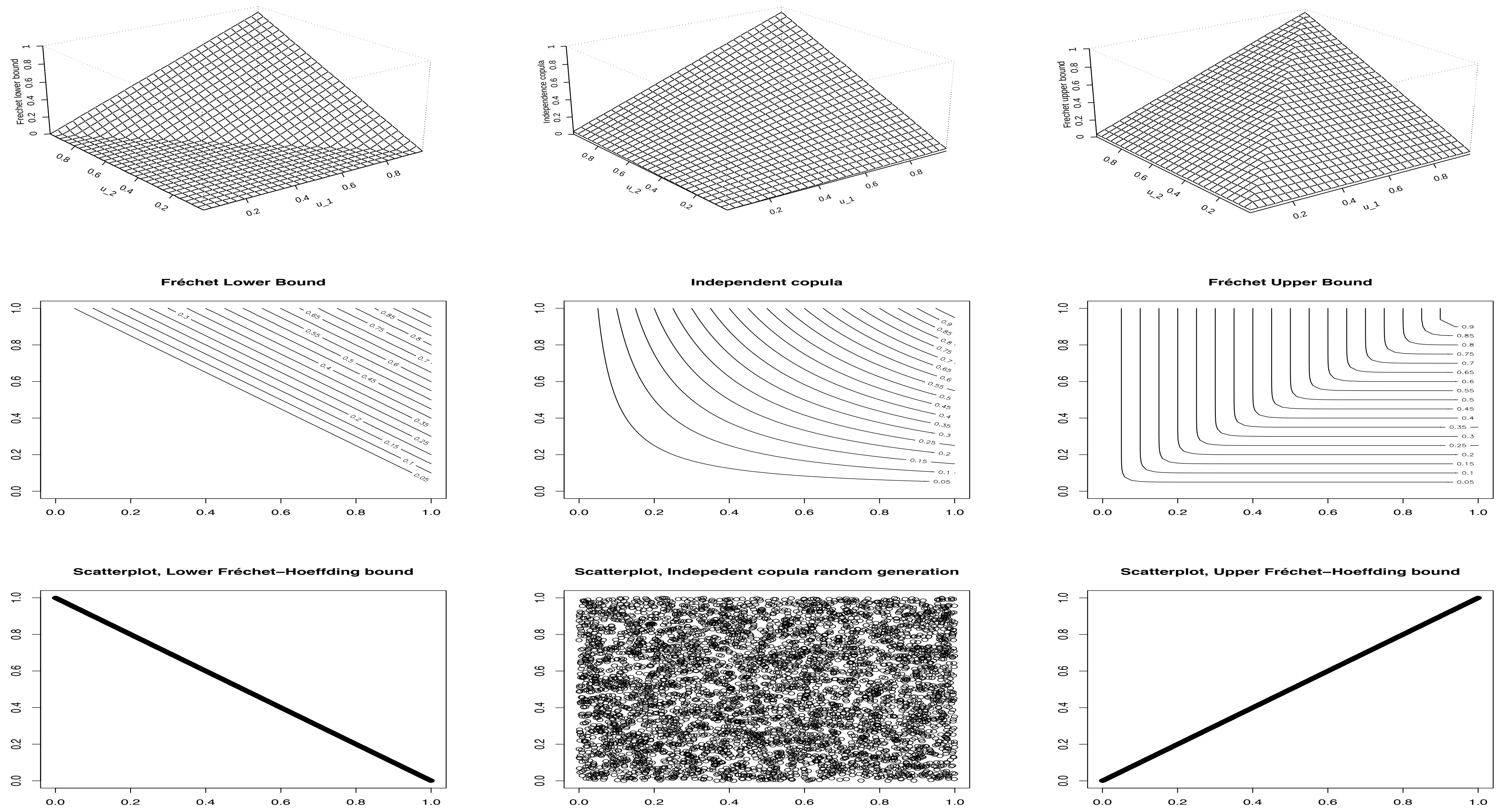

Fig. 5 – Contercomontonce, independent, and comonotone copulas.

Elliptical (Gaussian and t) copulas

The idea is to extend the multivariate probit model, $\boldsymbol{X} = (X_1, \ldots, X_d)$ with marginal $\mathcal{B}(p_i)$ distributions, modeled as $Y_i = \mathbf{1}(X_i^\star \leq u_i)$, where $\boldsymbol{X}^\star \sim \mathcal{N}(\mathbb{I}, \boldsymbol{\Sigma})$.

- The Gaussian copula, with parameter $\alpha \in (-1, 1)$,

$$C(u,v) = \frac{1}{2\pi\sqrt{1-\alpha^2}} \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \exp\left\{\frac{-(x^2 - 2\alpha xy + y^2)}{2(1-\alpha^2)}\right\} dxdy.$$

Analogously the t-copula is the distribution of $(T(X), T(Y))$ where T is the t-cdf, and where (X, Y) has a joint t-distribution.

- The Student t-copula with parameter $\alpha \in (-1, 1)$ and $\nu \geq 2$,

$$C(u,v) = \frac{1}{2\pi\sqrt{1-\alpha^2}} \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \left(1 + \frac{x^2 - 2\alpha xy + y^2}{2(1-\alpha^2)}\right)^{-((\nu+2)/2)} dxdy.$$

Archimedean copulas

- Archimedian copulas $C(u,v) = \phi^{-1}(\phi(u) + \phi(v))$, where ϕ is decreasing convex $(0,1)$, with $\phi(0) = \infty$ and $\phi(1) = 0$.

Example 3. *If $\phi(t) = [-\log t]^\alpha$, then C is Gumbel's copula, and if $\phi(t) = t^{-\alpha} - 1$, C is Clayton's. Note that $C^\perp$ is obtained when $\phi(t) = -\log t$.*

The frailty approach : assume that X and Y are conditionally independent, given the value of an heterogeneous component Θ. Assume further that

$$\mathbb{P}(X \leq x|\Theta = \theta) = (G_X(x))^\theta \text{ and } \mathbb{P}(Y \leq y|\Theta = \theta) = (G_Y(y))^\theta$$

for some baseline distribution functions G_X and G_Y. Then

$$F(x,y) = \psi(\psi^{-1}(F_X(x)) + \psi^{-1}(F_Y(y))),$$

where ψ denotes the Laplace transform of Θ, i.e. $\psi(t) = \mathbb{E}(e^{-t\Theta})$.

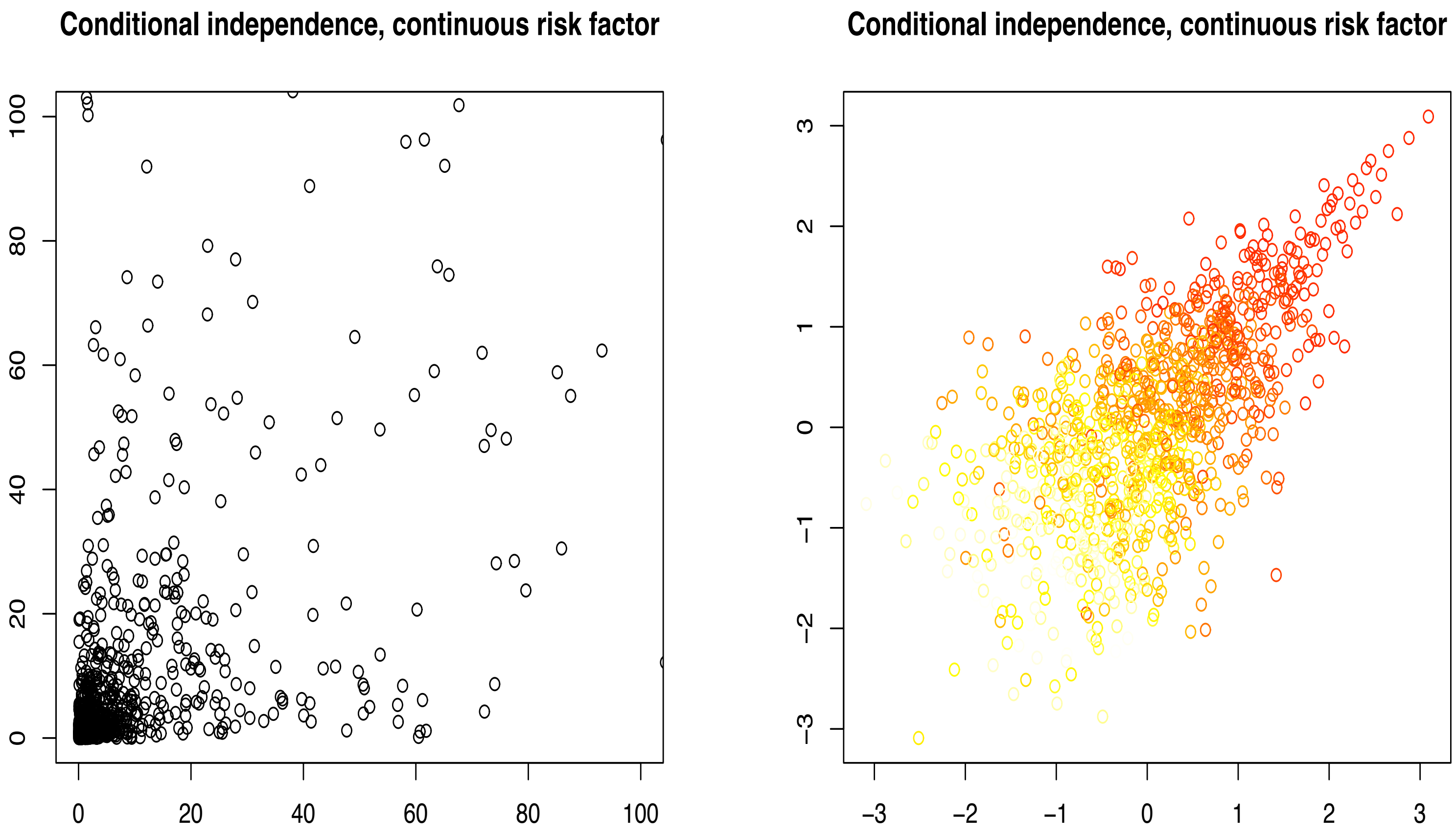

Fig. 6 – Continuous classes of risks, (X_i, Y_i) and $(\Phi^{-1}(F_X(X_i)), \Phi^{-1}(F_Y(Y_i)))$.

Some more examples of Archimedean copulas

	$\psi(t)$	range θ	
(1)	$\frac{1}{\theta}(t^{-\theta}-1)$	$[-1,0)\cup(0,\infty)$	Clayton, Clayton (1978)
(2)	$(1-t)^{\theta}$	$[1,\infty)$	
(3)	$\log\frac{1-\theta(1-t)}{t}$	$[-1,1)$	Ali-Mikhail-Haq
(4)	$(-\log t)^{\theta}$	$[1,\infty)$	Gumbel, Gumbel (1960), Hougaard (1986)
(5)	$-\log\frac{e^{-\theta t}-1}{e^{-\theta}-1}$	$(-\infty,0)\cup(0,\infty)$	Frank, Frank (1979), Nelsen (1987)
(6)	$-\log\{1-(1-t)^{\theta}\}$	$[1,\infty)$	Joe, Frank (1981), Joe (1993)
(7)	$-\log\{\theta t+(1-\theta)\}$	$(0,1]$	
(8)	$\frac{1-t}{1+(\theta-1)t}$	$[1,\infty)$	
(9)	$\log(1-\theta\log t)$	$(0,1]$	Barnett (1980), Gumbel (1960)
(10)	$\log(2t^{-\theta}-1)$	$(0,1]$	
(11)	$\log(2-t^{\theta})$	$(0,1/2]$	
(12)	$(\frac{1}{t}-1)^{\theta}$	$[1,\infty)$	
(13)	$(1-\log t)^{\theta}-1$	$(0,\infty)$	
(14)	$(t^{-1/\theta}-1)^{\theta}$	$[1,\infty)$	
(15)	$(1-t^{1/\theta})^{\theta}$	$[1,\infty)$	Genest & Ghoudi (1994)
(16)	$(\frac{\theta}{t}+1)(1-t)$	$[0,\infty)$	

Extreme value copulas

- Extreme value copulas

$$C(u,v)=\exp\left[(\log u+\log v)\,A\left(\frac{\log u}{\log u+\log v}\right)\right],$$

where A is a dependence function, convex on $[0,1]$ with $A(0)=A(1)=1$, et

$$\max\{1-\omega,\omega\}\leq A\left(\omega\right)\leq 1 \text{ for all } \omega\in\left[0,1\right].$$

An alternative definition is the following : C is an extreme value copula if for all $z>0$,

$$C(u_1,\ldots,u_d)=C(u_1^{1/z},\ldots,u_d^{1/z})^z.$$

Those copula are then called max-stable : define the maximum componentwise of a sample $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$, i.e. $M_i=\max\{X_{i,1},\ldots,X_{i,n}\}$.

Remark more difficult to characterize when $d\geq 3$.

On copula parametrization

- Gaussian, Student t (and elliptical) copulas

Focuses on pairwise dependence through the correlation matrix,

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{matrix} 1 & r_{12} & r_{13} & r_{14} \\ r_{12} & 1 & r_{23} & r_{24} \\ r_{13} & r_{23} & 1 & r_{34} \\ r_{14} & r_{24} & r_{34} & 1 \end{matrix} \right)$$

Dependence in $[0,1]^d \longleftrightarrow$ summarized in $d(d+1)/2$ parameters,

On copula parametrization

- Archimedean copulas

Initially, dependence in $[0,1]^d \longleftrightarrow$ summarized in one functional parameters on $[0,1]$. But appears less flexible because of exchangeability features.

$$\text{Let } \boldsymbol{U} = (U_1, U_2, U_3, U_4),$$

$$C(u_1, u_2, u_3, u_4) = \phi_1^{-1}[\phi_1(u_1) + \phi_1(u_2) + \phi_1(u_3) + \phi_1(u_4)],$$

On copula parametrization

- Archimedean copulas

Initially, dependence in $[0,1]^d \longleftrightarrow$ summarized in one functional parameters on $[0,1]$. But appears less flexible because of exchangeability features.

It is possible to introduce hierarchical Archimedean copulas (see Savu & Trede (2006) or McNeil (2007)). Let $\boldsymbol{U} = (U_1, U_2, U_3, U_4)$,

$$\phi_2^{-1}(\phi_2(u_1) + \phi_2(u_2))$$

On copula parametrization

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$$\phi_2^{-1}(\phi_2(u_1) + \phi_2(u_2)) \qquad\qquad \phi_3^{-1}(\phi_3(u_3) + \phi_3(u_4))$$

On copula parametrization

- Archimedean copulas

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It is possible to introduce hierarchical Archimedean copulas (see Savu & Trede (2006) or McNeil (2007)). Let $\boldsymbol{U} = (U_1, U_2, U_3, U_4)$,

$$C(u_1,u_2,u_3,u_4) = \phi_4^{-1}(\phi_4\left[\phi_2^{-1}(\phi_2(u_1)+\phi_2(u_2))\right] + \phi_4\left[\phi_3^{-1}(\phi_3(u_3)+\phi_3(u_4))\right]),$$

which, if ϕ_i is parametrized with parameter α_i, can be summarized through

$$A = \left(\begin{array}{cc|cc} 1 & \alpha_2 & \alpha_4 & \alpha_4 \\ \alpha_2 & 1 & \alpha_4 & \alpha_4 \\ \hline \alpha_4 & \alpha_4 & 1 & \alpha_3 \\ \alpha_4 & \alpha_4 & \alpha_3 & 1 \end{array}\right)$$

On copula parametrization

- Archimedean copulas

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$$\phi_2^{-1}(\phi_2(u_1) + \phi_2(u_2))$$

On copula parametrization

- Archimedean copulas

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It is possible to introduce hierarchical Archimedean copulas (see Savu & Trede (2006) or McNeil (2007)). Let $\boldsymbol{U} = (U_1, U_2, U_3, U_4)$,

$$\phi_3^{-1}(\phi_3\left[\phi_2^{-1}(\phi_2(u_1) + \phi_2(u_2))\right] + \phi_3(u_3))$$

On copula parametrization

- Archimedean copulas

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It is possible to introduce hierarchical Archimedean copulas (see Savu & Trede (2006) or McNeil (2007)). Let $\boldsymbol{U} = (U_1, U_2, U_3, U_4)$,

$$C(u_1,u_2,u_3,u_4) = \phi_4^{-1}(\phi_4[\phi_3^{-1}(\phi_3\left[\phi_2^{-1}(\phi_2(u_1)+\phi_2(u_2))\right]+\phi_3(u_3))]+\phi_4(u_4)),$$

which, if ϕ_i is parametrized with parameter α_i, can be summarized through

$$A = \begin{pmatrix} 1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & 1 & \alpha_3 & \alpha_4 \\ \alpha_3 & \alpha_3 & 1 & \alpha_4 \\ \alpha_4 & \alpha_4 & \alpha_4 & 1 \end{pmatrix}$$

On copula parametrization

- Extreme value copulas

Here, dependence in $[0,1]^d \longleftrightarrow$ summarized in one functional parameters on $[0,1]^{d-1}$.

Further, focuses only on first order tail dependence.

Natural properties for dependence measures

Definition 4. *κ is measure of concordance if and only if κ satisfies*

- *κ is defined for every pair (X, Y) of continuous random variables,*
- $-1 \leq \kappa(X, Y) \leq +1$, $\kappa(X, X) = +1$ *and* $\kappa(X, -X) = -1$,
- $\kappa(X, Y) = \kappa(Y, X)$,
- *if X and Y are independent, then $\kappa(X, Y) = 0$,*
- $\kappa(-X, Y) = \kappa(X, -Y) = -\kappa(X, Y)$,
- *if $(X_1, Y_1) \preceq_{PQD} (X_2, Y_2)$, then $\kappa(X_1, Y_1) \leq \kappa(X_2, Y_2)$,*
- *if $(X_1, Y_1), (X_2, Y_2), \ldots$ is a sequence of continuous random vectors that converge to a pair (X, Y) then $\kappa(X_n, Y_n) \to \kappa(X, Y)$ as $n \to \infty$.*

Natural properties for dependence measures

If κ is measure of concordance, then, if f and g are both strictly increasing, then $\kappa(f(X), g(Y)) = \kappa(X, Y)$. Further, $\kappa(X, Y) = 1$ if $Y = f(X)$ with f almost surely strictly increasing, and analogously $\kappa(X, Y) = -1$ if $Y = f(X)$ with f almost surely strictly decreasing (see Scarsini (1984)).

Rank correlations can be considered, i.e. Spearman's ρ defined as

$$\rho(X, Y) = corr(F_X(X), F_Y(Y)) = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3$$

and Kendall's τ defined as

$$\tau(X, Y) = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1.$$

Historical version of those coefficients

Similarly Kendall's tau was not defined using copulae, but as the probability of concordance, minus the probability of discordance, i.e.

$$\tau(X,Y) = 3[\mathbb{P}((X_1 - X_2)(Y_1 - Y_2) > 0) - \mathbb{P}((X_1 - X_2)(Y_1 - Y_2) < 0)],$$

where (X_1, Y_1) and (X_2, Y_2) denote two independent versions of (X, Y) (see Nelsen (1999)).

Equivalently, $\tau(X,Y) = 1 - \dfrac{4Q}{n(n^2-1)}$ where Q is the number of inversions between the rankings of X and Y (number of discordance).

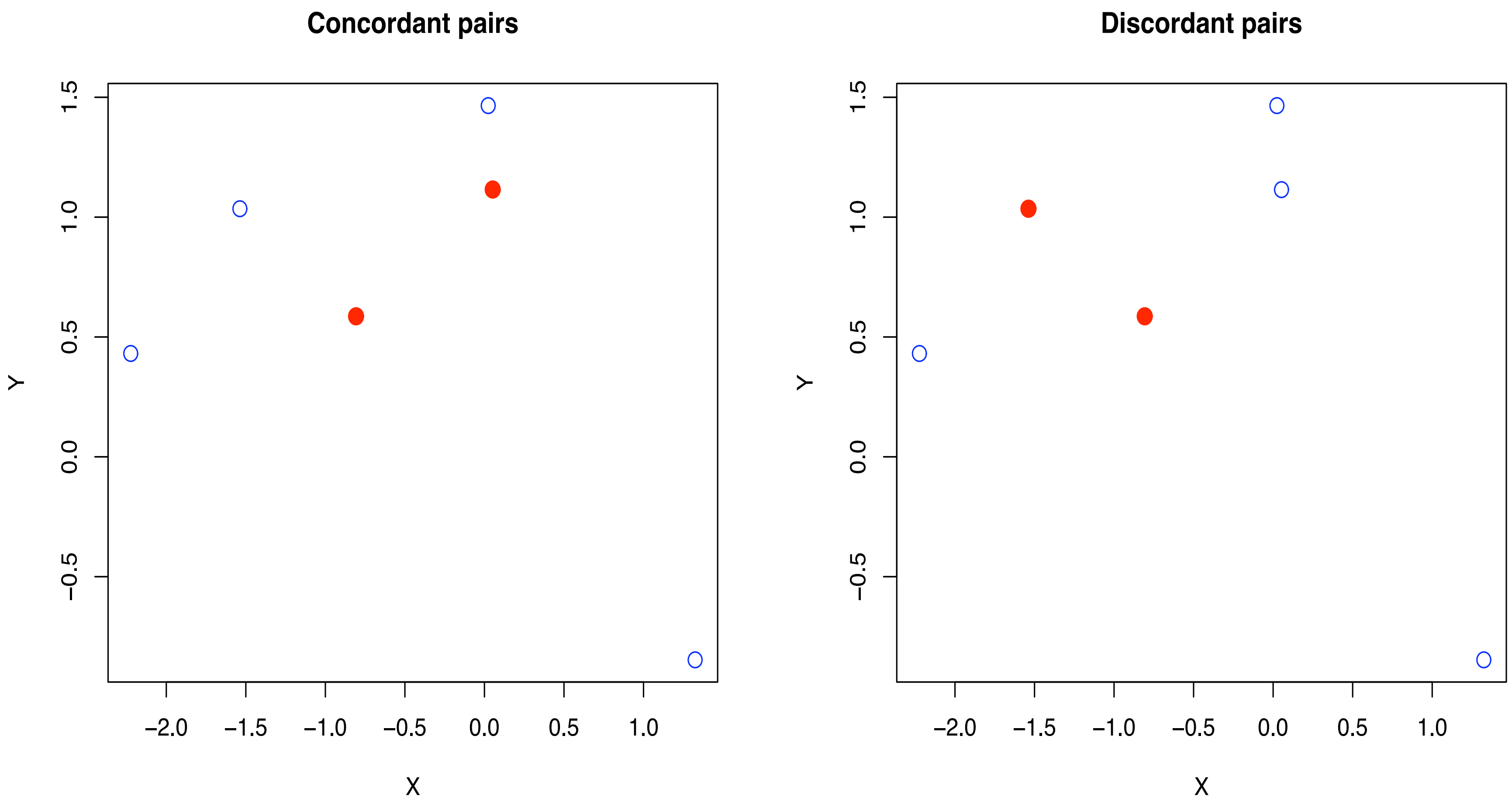

Fig. 7 – Concordance versus discordance.

Alternative expressions of those coefficients

Note that those coefficients can also be expressed as follows

$$\rho(X,Y)=\frac{\int_{[0,1]\times[0,1]}C(u,v)-C^{\perp}(u,v)dudv}{\int_{[0,1]\times[0,1]}C^{+}(u,v)-C^{\perp}(u,v)dudv}$$

(the normalized average distance between C and $C^{\perp}$), for instance.

The case of the Gaussian random vector

If (X,Y) is a Gaussian random vector with correlation r, then (Kruskal (1958))

$$\rho(X,Y)=\frac{6}{\pi}\arcsin\left(\frac{r}{2}\right)\text{ and }\tau(X,Y)=\frac{2}{\pi}\arcsin\left(r\right).$$

From Kendall'tau to copula parameters

Kendall's τ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Gaussian θ	0.00	0.16	0.31	0.45	0.59	0.71	0.81	0.89	0.95	0.99	1.00
Gumbel θ	1.00	1.11	1.25	1.43	1.67	2.00	2.50	3.33	5.00	10.0	$+\infty$
Plackett θ	1.00	1.57	2.48	4.00	6.60	11.4	21.1	44.1	115	530	$+\infty$
Clayton θ	0.00	0.22	0.50	0.86	1.33	2.00	3.00	4.67	8.00	18.0	$+\infty$
Frank θ	0.00	0.91	1.86	2.92	4.16	5.74	7.93	11.4	18.2	20.9	$+\infty$
Joe θ	1.00	1.19	1.44	1.77	2.21	2.86	3.83	4.56	8.77	14.4	$+\infty$
Galambos θ	0.00	0.34	0.51	0.70	0.95	1.28	1.79	2.62	4.29	9.30	$+\infty$
Morgenstein θ	0.00	0.45	0.90	-	-	-	-	-	-	-	-

From Spearman's rho to copula parameters

Spearman's	ρ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Gaussian	θ	0.00	0.10	0.21	0.31	0.42	0.52	0.62	0.72	0.81	0.91	1.00
Gumbel	θ	1.00	1.07	1.16	1.26	1.38	1.54	1.75	2.07	2.58	3.73	$+\infty$
A.M.H.	θ	1.00	1.11	1.25	1.43	1.67	2.00	2.50	3.33	5.00	10.0	$+\infty$
Plackett	θ	1.00	1.35	1.84	2.52	3.54	5.12	7.76	12.7	24.2	66.1	$+\infty$
Clayton	θ	0.00	0.14	0.31	0.51	0.76	1.06	1.51	2.14	3.19	5.56	$+\infty$
Frank	θ	0.00	0.60	1.22	1.88	2.61	3.45	4.47	5.82	7.90	12.2	$+\infty$
Joe	θ	1.00	1.12	1.27	1.46	1.69	1.99	2.39	3.00	4.03	6.37	$+\infty$
Galambos	θ	0.00	0.28	0.40	0.51	0.65	0.81	1.03	1.34	1.86	3.01	$+\infty$
Morgenstein	θ	0.00	0.30	0.60	0.90	-	-	-	-	-	-	-

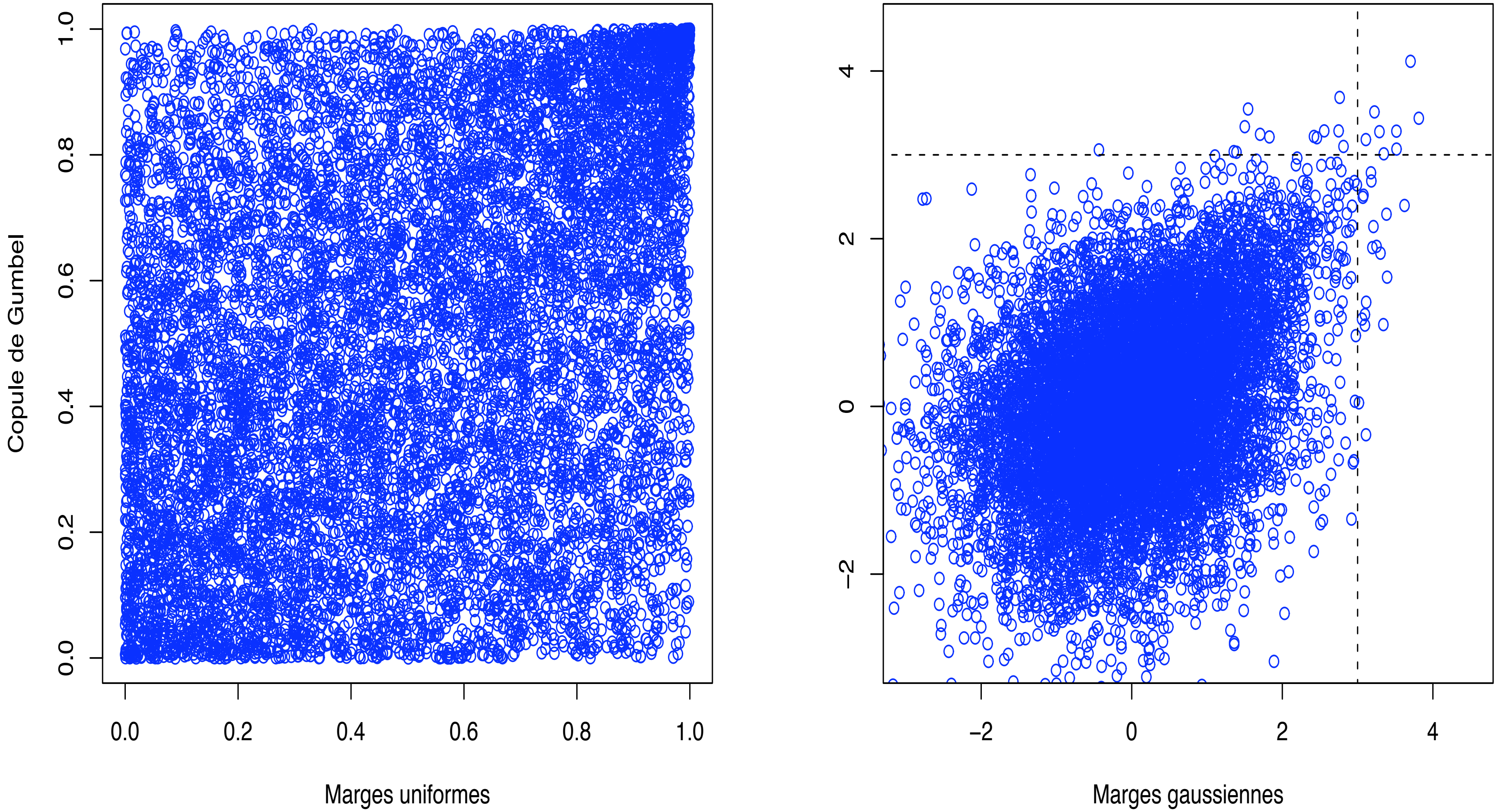

Fig. 8 – Simulations of Gumbel's copula $\theta = 1.2$.

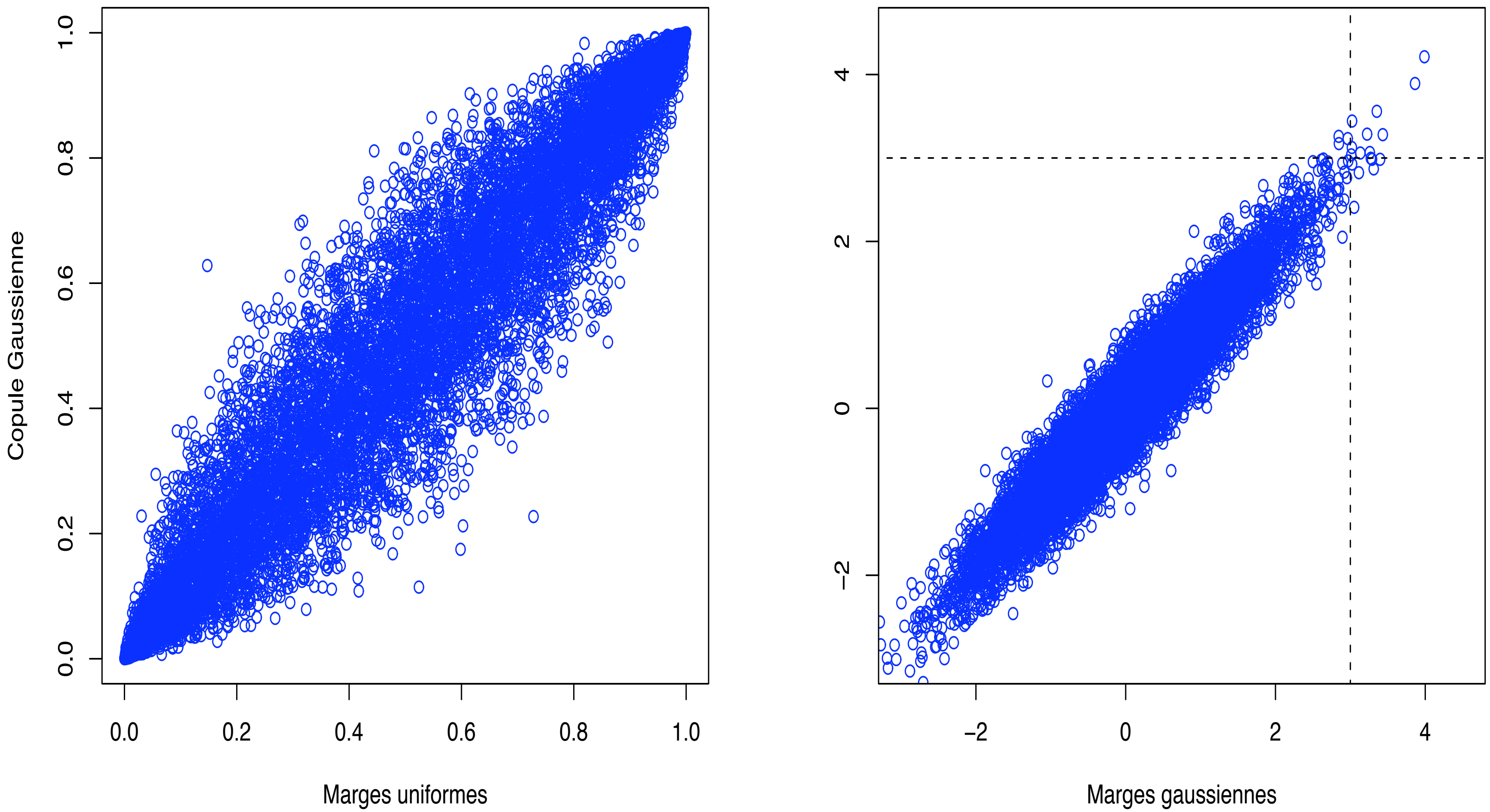

Fig. 9 – Simulations of the Gaussian copula ($\theta = 0.95$).

Tail correlation and Solvency II

4. Copulas versus the use of Tail correlation factors

The IAA proposes to use Copulas as the theoretically correct method to calculate diversification effects. Indeed the use of a "standard" correlation matrix is wrong. Copulas have the advantage that they can be used to accurately combine other distributions than from the "Normal Family" and that they can recognise dependencies that change in the tail of the distribution.

Severe incidents can impact risks that are normally independent. Example: normally market risk and mortality risk will be independent. But when a severe pandemic like the Spanish Flu would happen with world-wide millions of deaths this will certainly have economic consequences and will also impact market risk (for example equity-risk).

In practice combining several distributions implies that the dependency in the tail is higher than on average.

A problem with the use of Copulas is that it is very complex in the case that a rather large number of distributions have to be combined. Also there is generally limited data available to estimate the copula function in the tail. Given these observations many practitioners consider that a simpler approach can deliver acceptable results.
A more detailed explanation of Copulas can be found in Appendix A

Tail correlation and Solvency II

5. Estimation of tail-correlation factors

The estimation of the correlation between two risks under extreme circumstances is subject to the same uncertainties as the selecting of copula functions. There will never be enough data for a reliable estimation. By definition extreme situations will not happen frequently. Extreme events that will happen in the future did not happen yet in the past. The only possibility we have is the use of scientific evidence on dependencies, based on semi-worse case events in the past and expert opinion and to get an agreement between industry partners and the regulators.

2- Measures of dependence.

It is important to use copulas to get a better understanding (and so a measure) of the kind of dependence that exists, especially in the tails (because its with the case of extreme outcomes that we must worry!) of the joint distribution function, which can be done through the tail dependence coefficient (λ). This method is preferable to using only the simple linear correlation, which plays a central role in financial theory (as can be seen in the CAPM), but which is only theoretically correct with elliptical distributions (distributions whose density is constant on ellipsoids), such as the Normal.

Strong tail dependence

Joe (1993) defined, in the bivariate case a tail dependence measure.

Definition 5. *Let (X, Y) denote a random pair, the upper and lower tail dependence parameters are defined, if the limit exist, as*

$$\begin{aligned}\lambda_L &= \lim_{u\to 0} \mathbb{P}\left(X \leq F_X^{-1}(u)\,|Y \leq F_Y^{-1}(u)\right), \\ &= \lim_{u\to 0} \mathbb{P}\left(U \leq u|V \leq u\right) = \lim_{u\to 0} \frac{C(u,u)}{u},\end{aligned}$$

and

$$\begin{aligned}\lambda_U &= \lim_{u\to 1} \mathbb{P}\left(X > F_X^{-1}(u)\,|Y > F_Y^{-1}(u)\right) \\ &= \lim_{u\to 0} \mathbb{P}\left(U > 1-u|V \leq 1-u\right) = \lim_{u\to 0} \frac{C^\star(u,u)}{u}.\end{aligned}$$

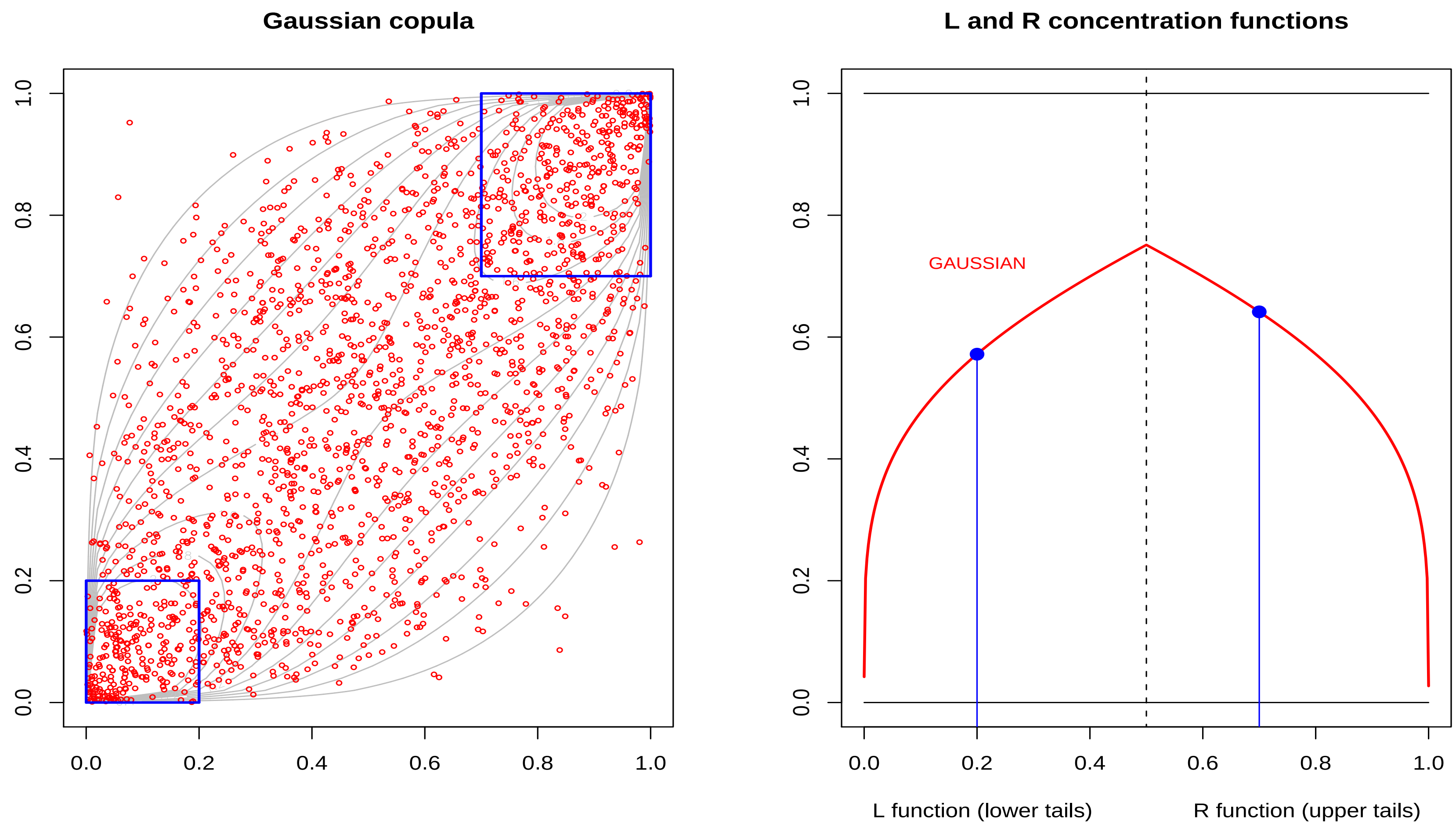

Fig. 10 – L and R cumulative curves.

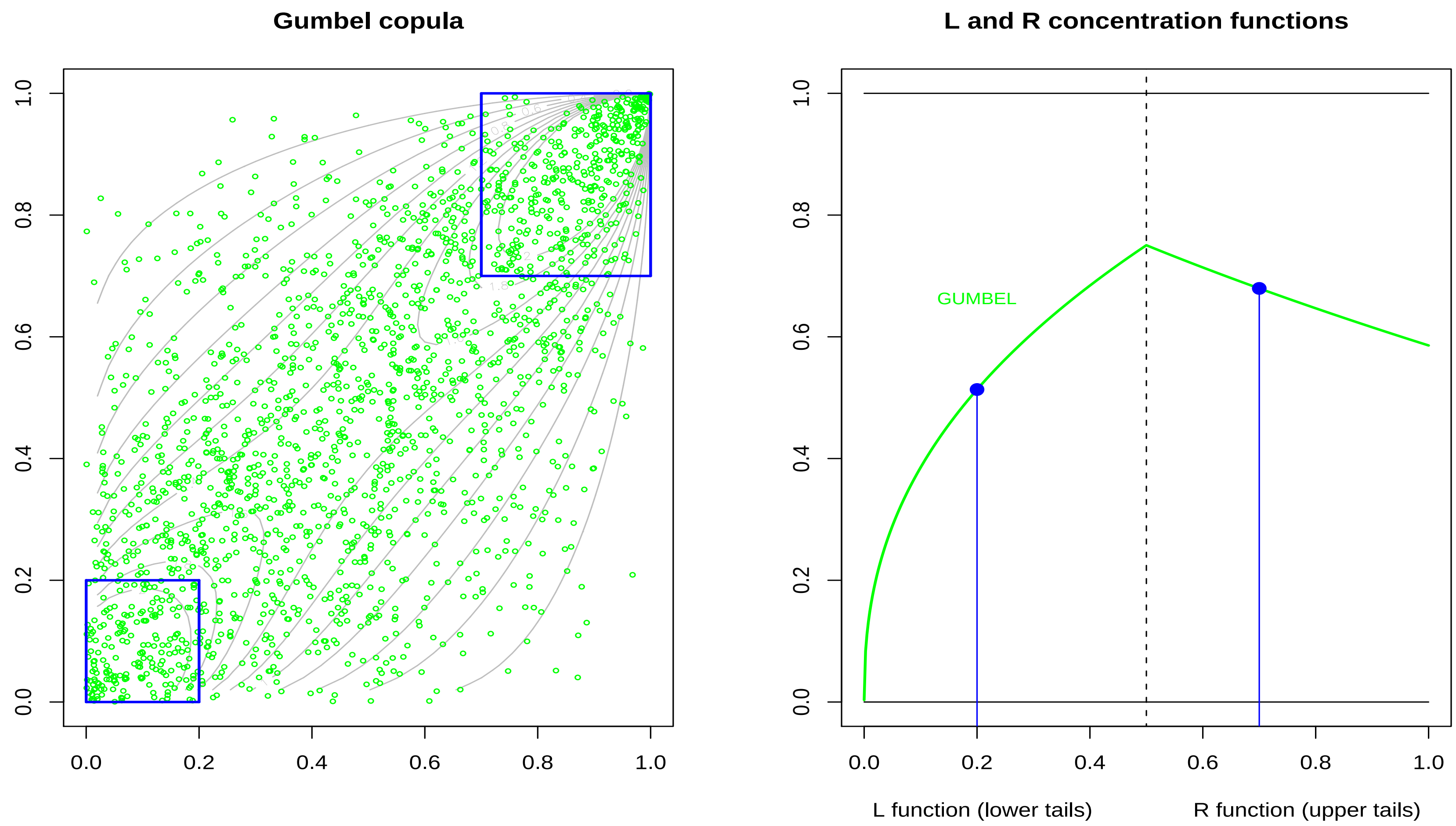

Fig. 11 – L and R cumulative curves.

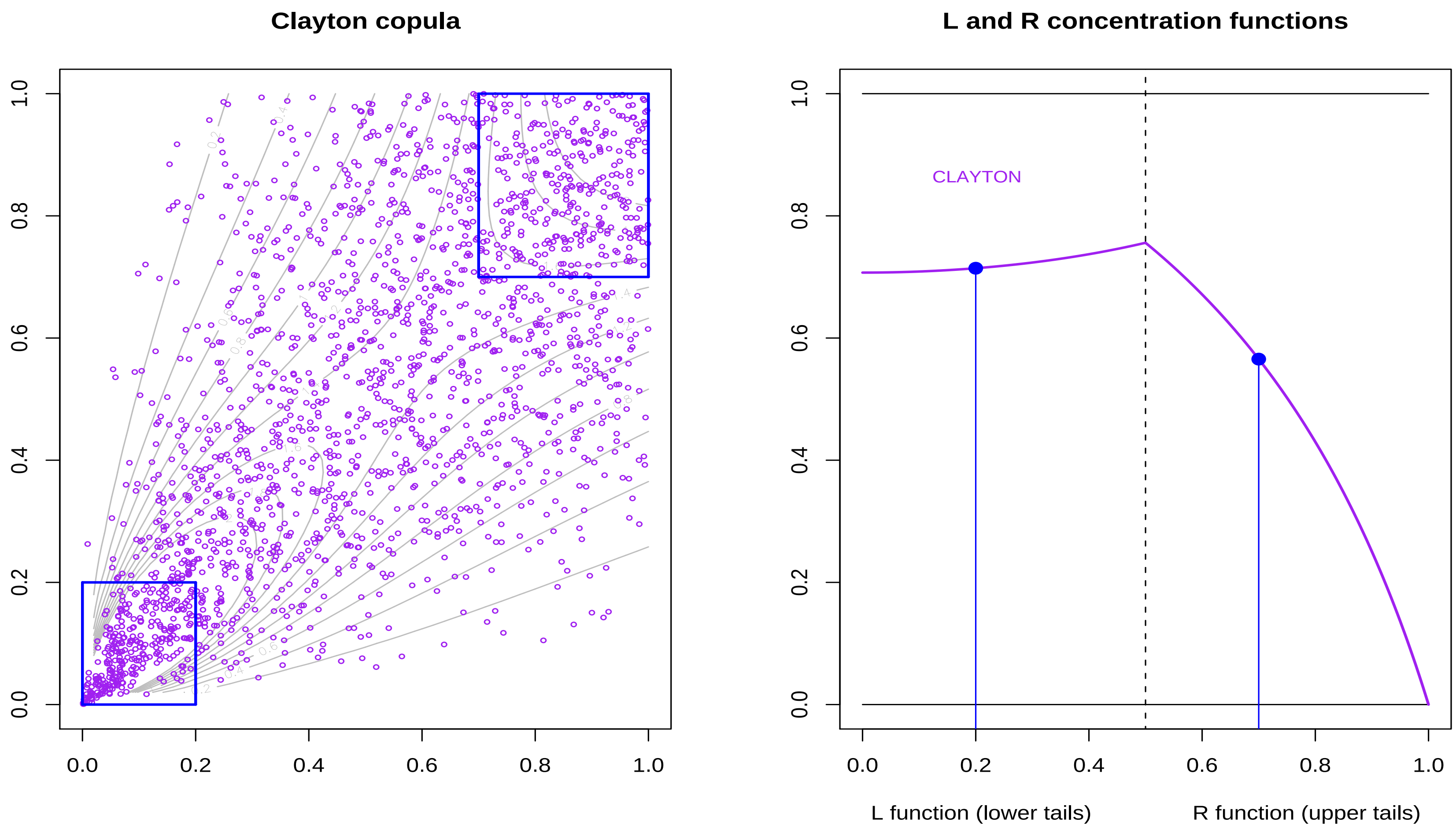

Fig. 12 – L and R cumulative curves.

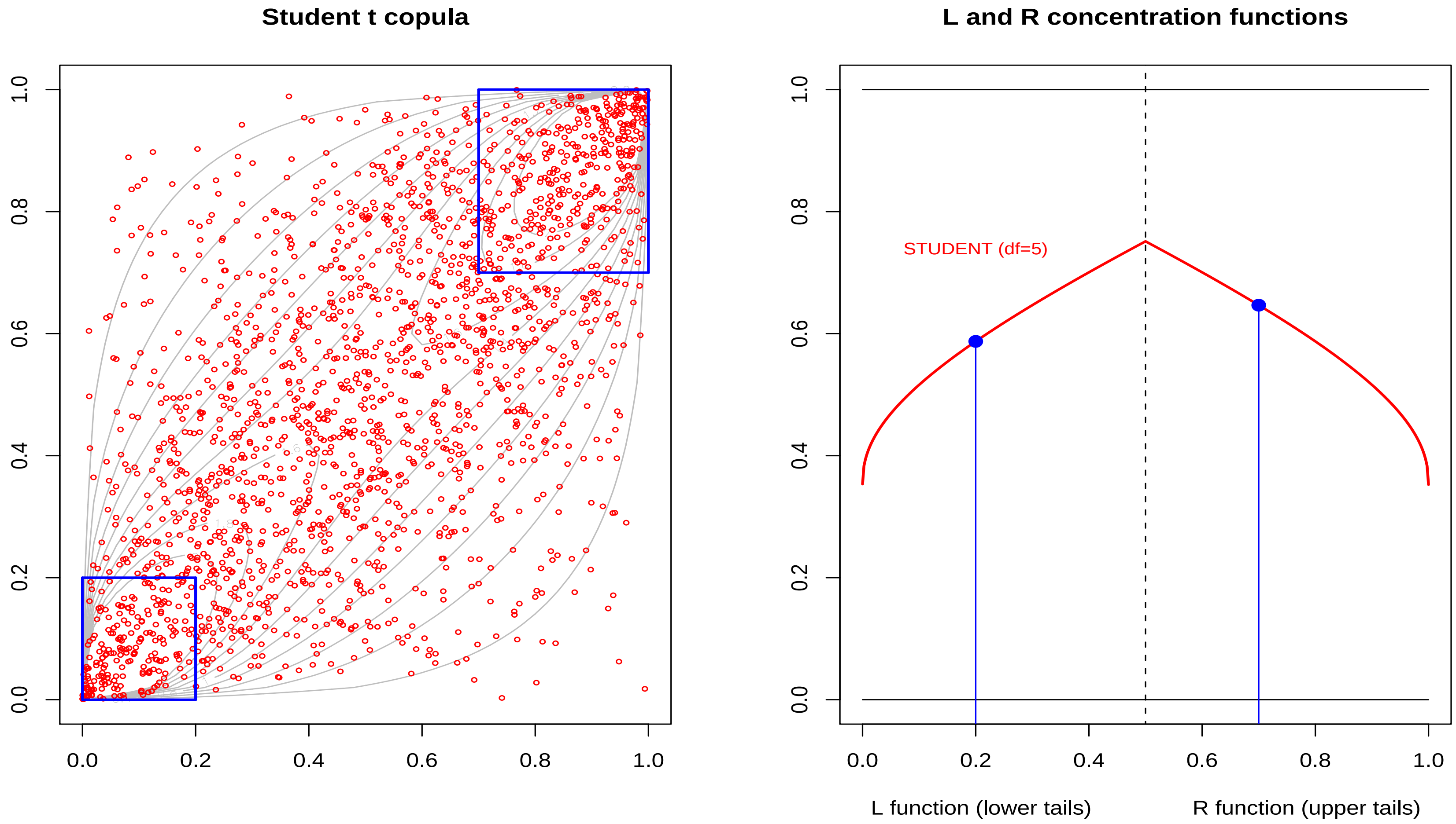

Fig. 13 – L and R cumulative curves.

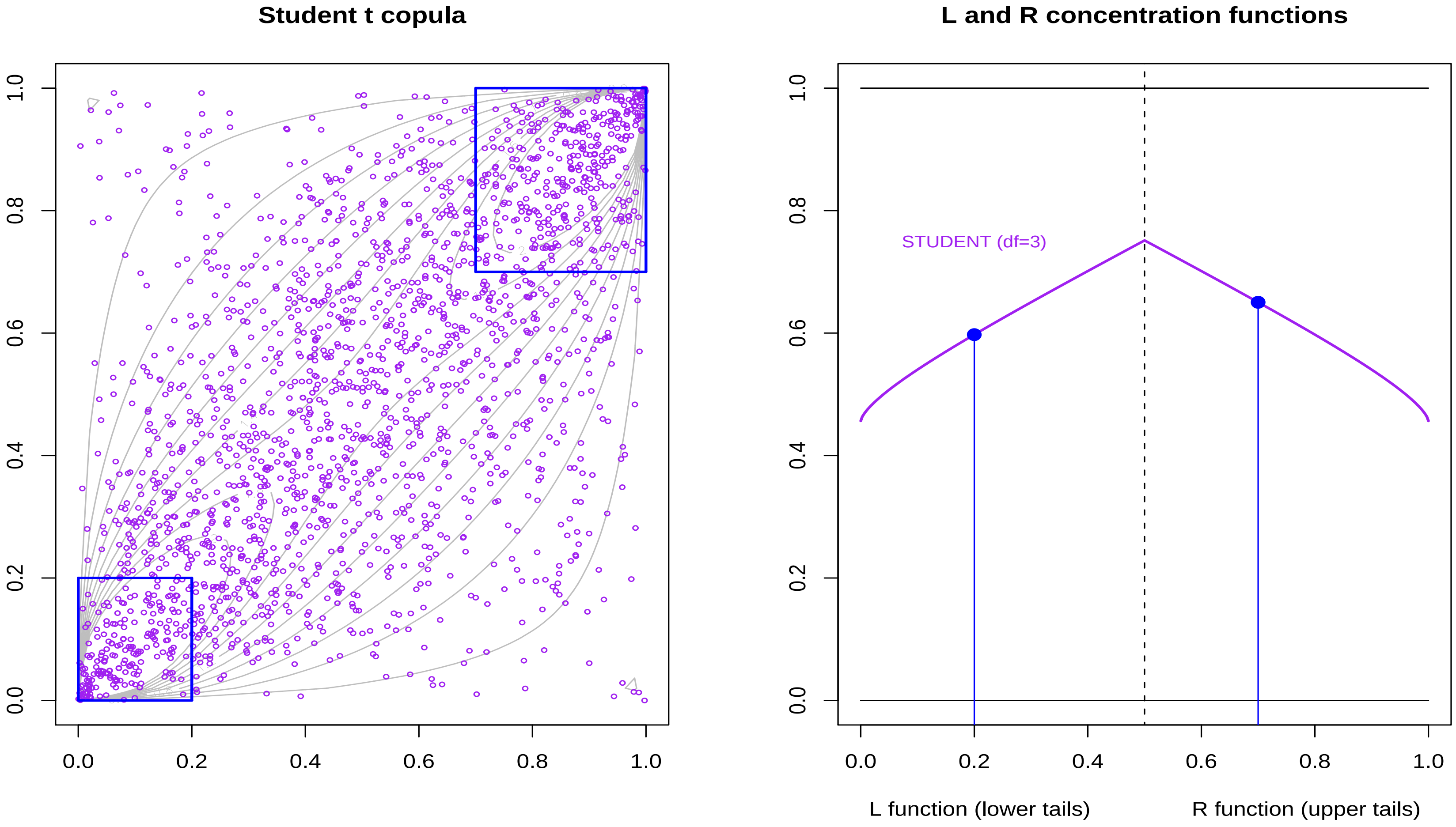

Fig. 14 – L and R cumulative curves.

Estimation of tail dependence

CEA
Insurers of Europe

Calibrating the Standard Approach: Building Blocks

Annex to ECO 7032

5 February 2007

Calibrating extreme events

Principle 8: *The assumptions for extreme tail events are likely to require an element of judgement*

In practice there is unlikely to be sufficient past data to set extreme tail values with confidence and hence this will require an element of judgement.

One method to estimate the tail calibration would be to fit a mathematical formula to the credible data, which might say be for the 25^{th} to 75^{th} percentiles and then to extrapolate using this formula.

An alternative approach might be to extrapolate using an assumed distribution (e.g. Normal, Lognormal, etc) for the point beyond which the past data has insufficient credibility. Different assumptions for this extrapolation process could result in significantly different extreme tail values. The impact of these assumptions and the quantum of uncertainty surrounding these assumptions need to be understood.

Estimating (strong) tail dependence

From

$$\mathbb{P} \approx \frac{\mathbb{P}\left(X > F_X^{-1}(u), Y > F_Y^{-1}(u)\right)}{\mathbb{P}\left(Y > F_Y^{-1}(u)\right)} \text{ for } u \text{ closed to } 1,$$

as for Hill's estimator, a *natural* estimator for λ is obtained with $u = 1 - k/n$,

$$\widehat{\lambda}_U^{(k)} = \frac{\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i > X_{n-k:n}, Y_i > Y_{n-k:n})}{\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(Y_i > Y_{n-k:n})},$$

hence

$$\widehat{\lambda}_U^{(k)} = \frac{1}{k}\sum_{i=1}^{n} \mathbf{1}(X_i > X_{n-k:n}, Y_i > Y_{n-k:n}).$$

$$\widehat{\lambda}_L^{(k)} = \frac{1}{k}\sum_{i=1}^{n} \mathbf{1}(X_i \leq X_{k:n}, Y_i \leq Y_{k:n}).$$

Asymptotic convergence, how fast ?

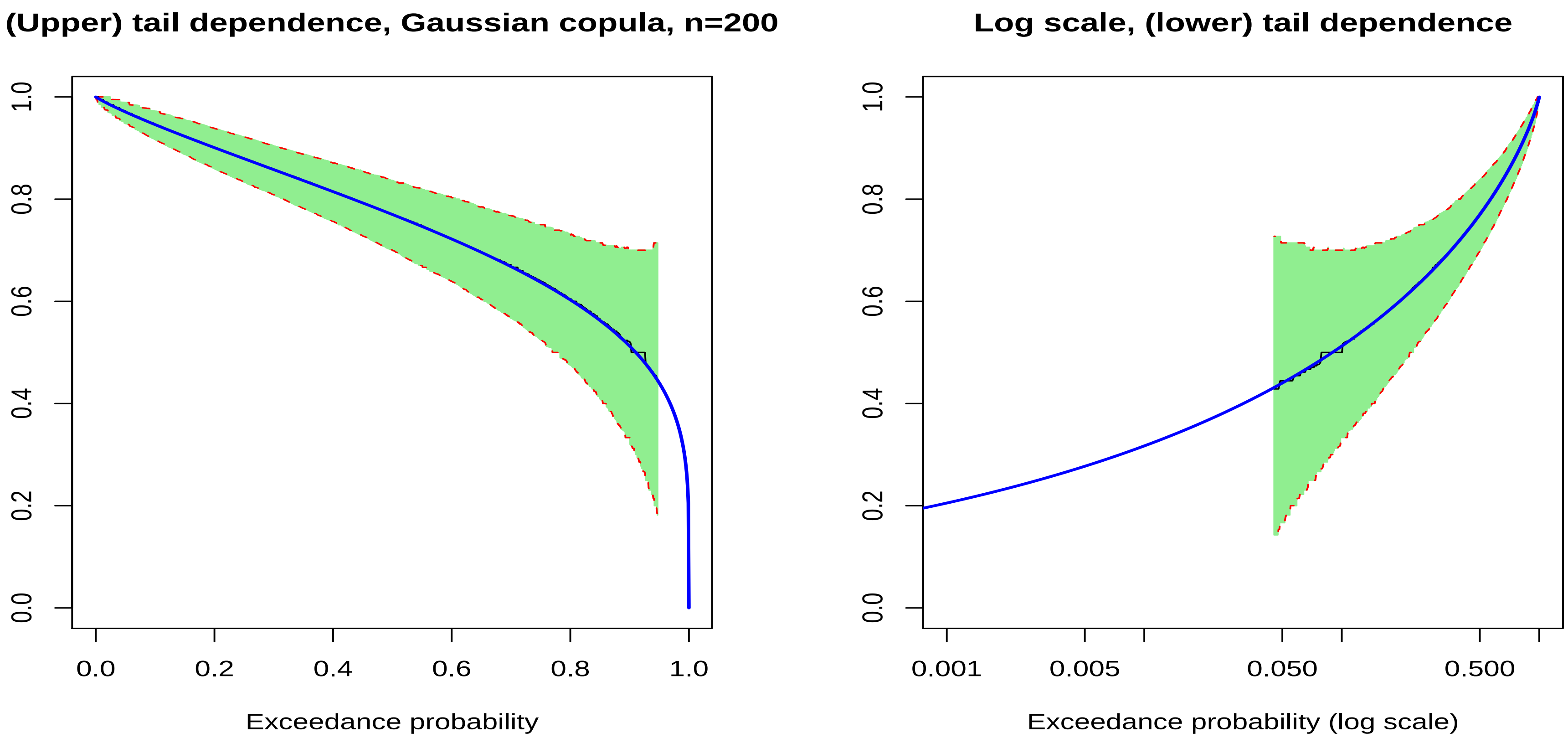

Fig. 15 – Convergence of L and R functions, Gaussian copula, $n = 200$.

Asymptotic convergence, how fast ?

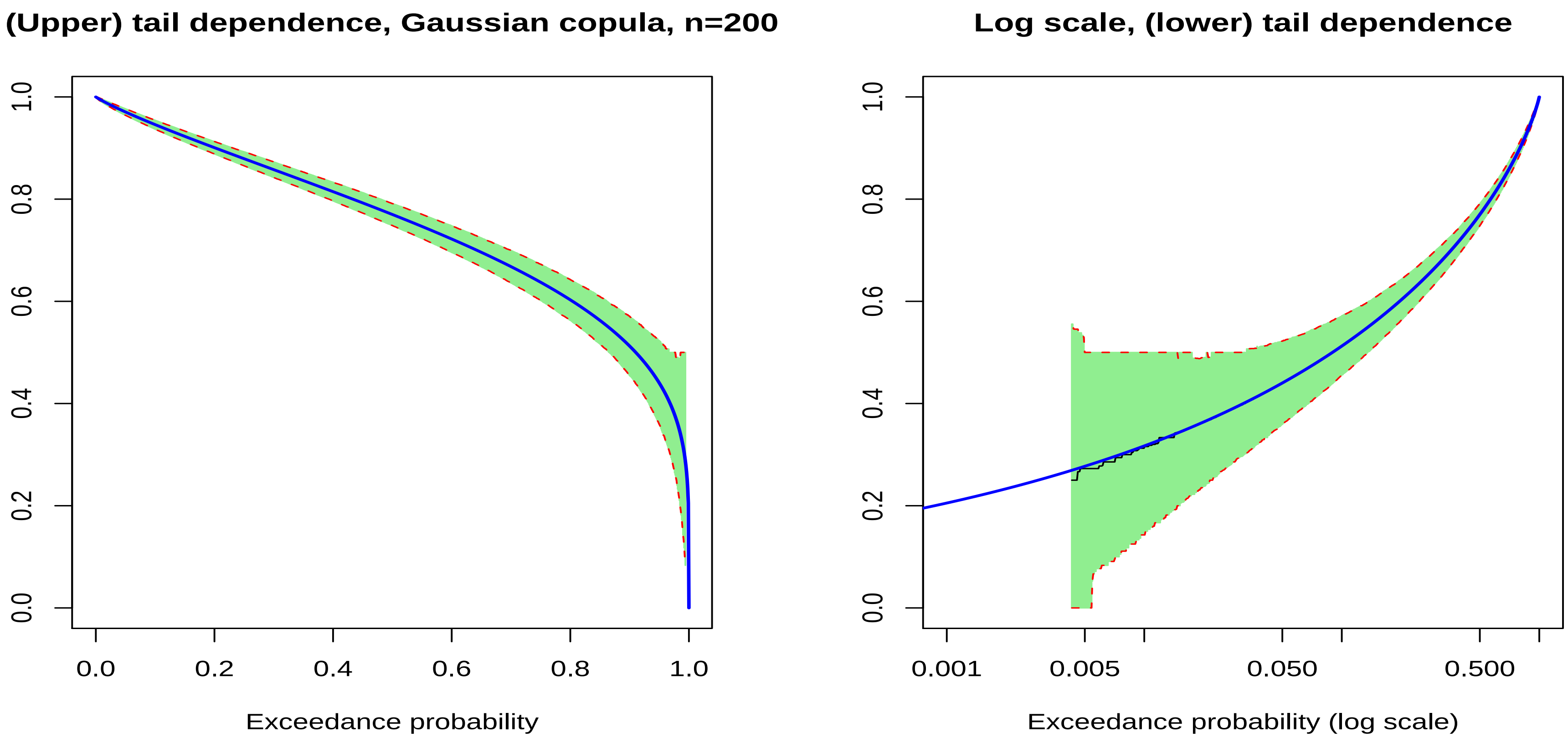

Fig. 16 – Convergence of L and R functions, Gaussian copula, $n = 2,000$.

Asymptotic convergence, how fast ?

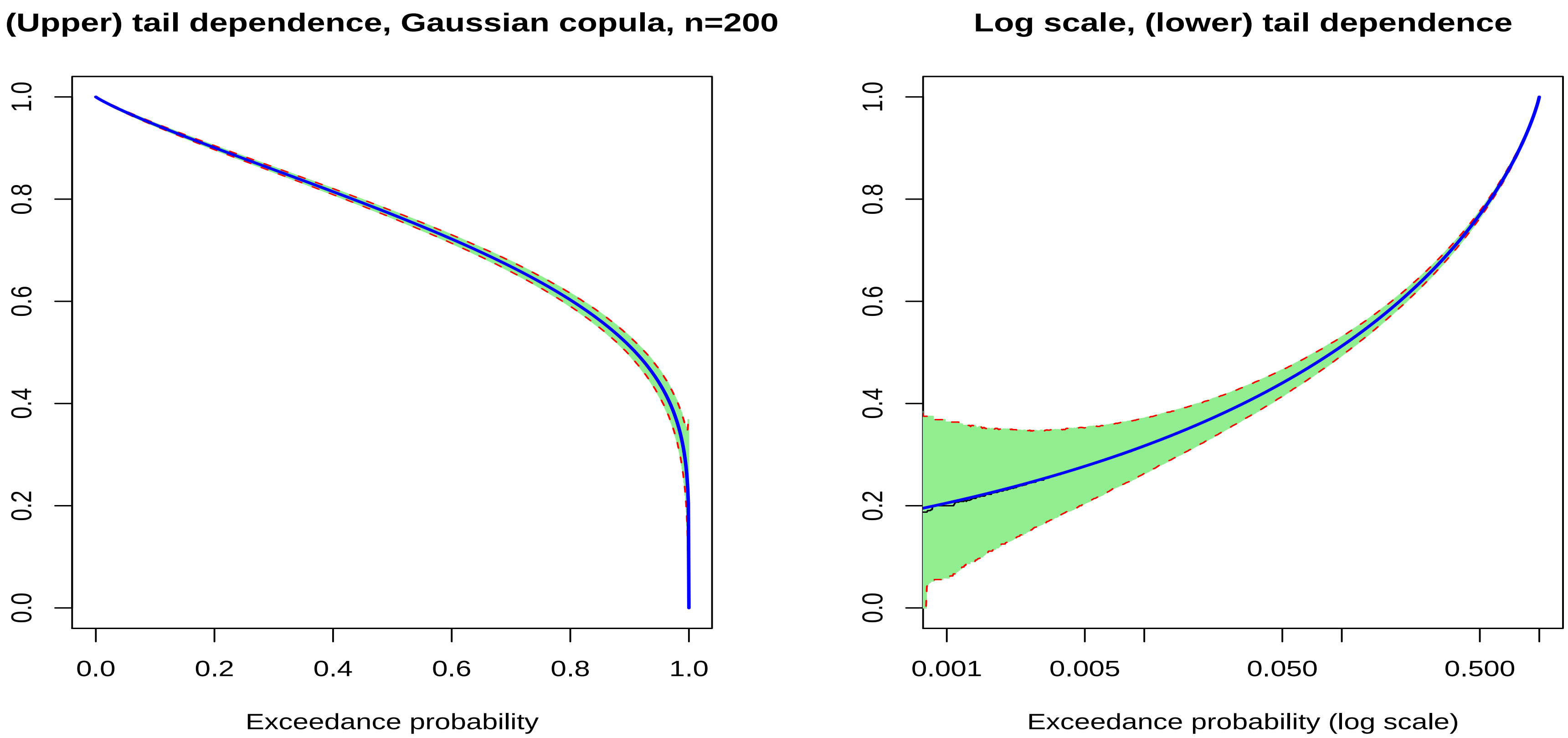

Fig. 17 – Convergence of L and R functions, Gaussian copula, $n = 20,000$.

Weak tail dependence

If X and Y are independent (in tails), for u large enough

$$\mathbb{P}(X > F_X^{-1}(u), Y > F_Y^{-1}(u)) = \mathbb{P}(X > F_X^{-1}(u)) \cdot \mathbb{P}(Y > F_Y^{-1}(u)) = (1-u)^{2},$$

or equivalently, $\log \mathbb{P}(X > F_X^{-1}(u), Y > F_Y^{-1}(u)) = 2 \cdot \log(1-u)$. Further, if X and Y are comonotonic (in tails), for u large enough

$$\mathbb{P}(X > F_X^{-1}(u), Y > F_Y^{-1}(u)) = \mathbb{P}(X > F_X^{-1}(u)) = (1-u)^{1},$$

or equivalently, $\log \mathbb{P}(X > F_X^{-1}(u), Y > F_Y^{-1}(u)) = 1 \cdot \log(1-u)$.

$\Longrightarrow$ limit of the ratio $\dfrac{\log(1-u)}{\log \mathbb{P}(Z_1 > F_1^{-1}(u), Z_2 > F_2^{-1}(u))}$.

Weak tail dependence

Coles, Heffernan & Tawn (1999) defined

Definition 6. *Let (X, Y) denote a random pair, the upper and lower tail dependence parameters are defined, if the limit exist, as*

$$\eta_L = \lim_{u \to 0} \frac{\log(u)}{\log \mathbb{P}(Z_1 \leq F_1^{-1}(u), Z_2 \leq F_2^{-1}(u))} = \lim_{u \to 0} \frac{\log(u)}{\log C(u,u)},$$

and

$$\eta_U = \lim_{u \to 1} \frac{\log(1-u)}{\log \mathbb{P}(Z_1 > F_1^{-1}(u), Z_2 > F_2^{-1}(u))} = \lim_{u \to 0} \frac{\log(u)}{\log C^{\star}(u,u)}.$$

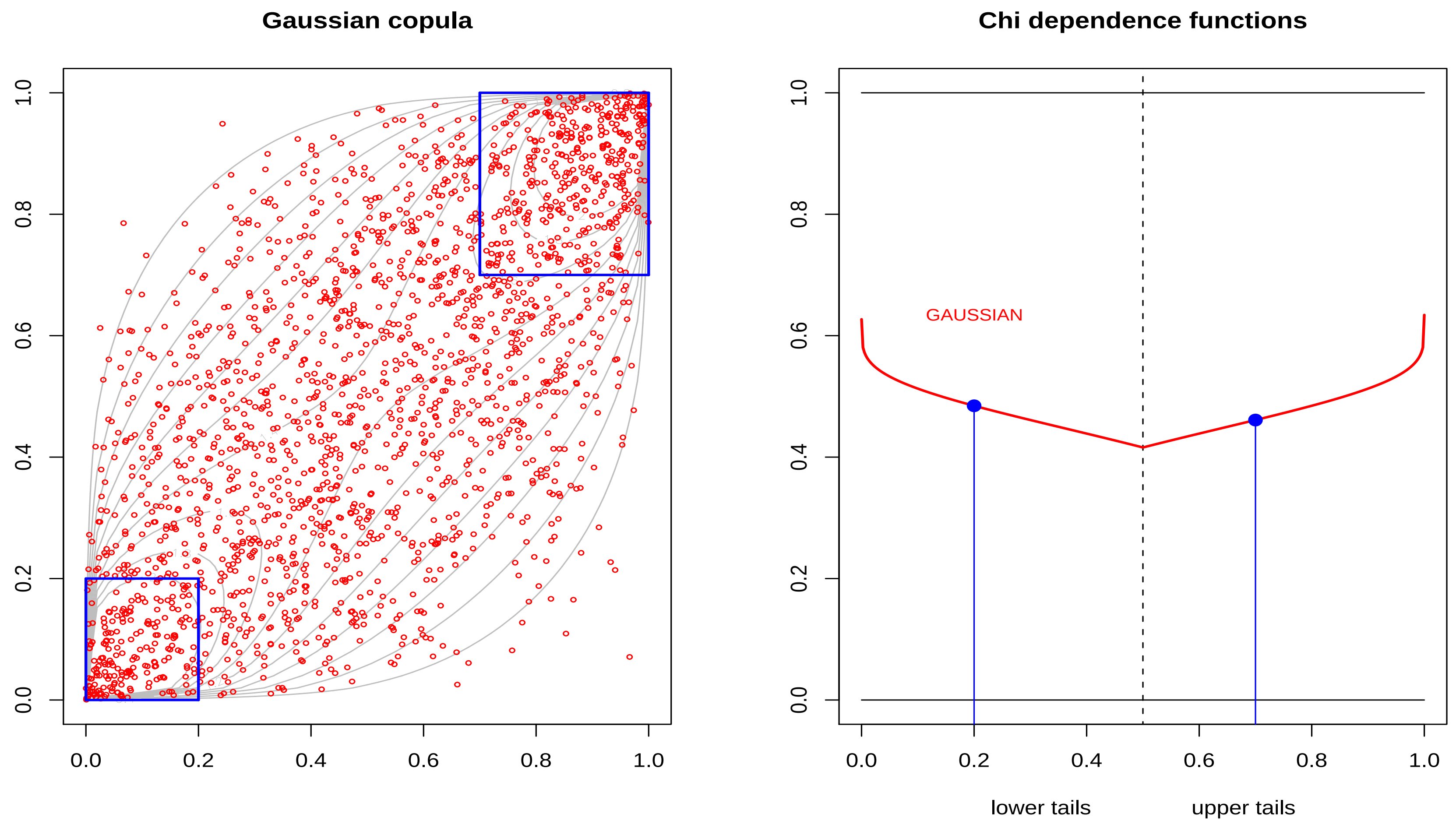

Fig. 18 – $\overline{\chi}$ functions.

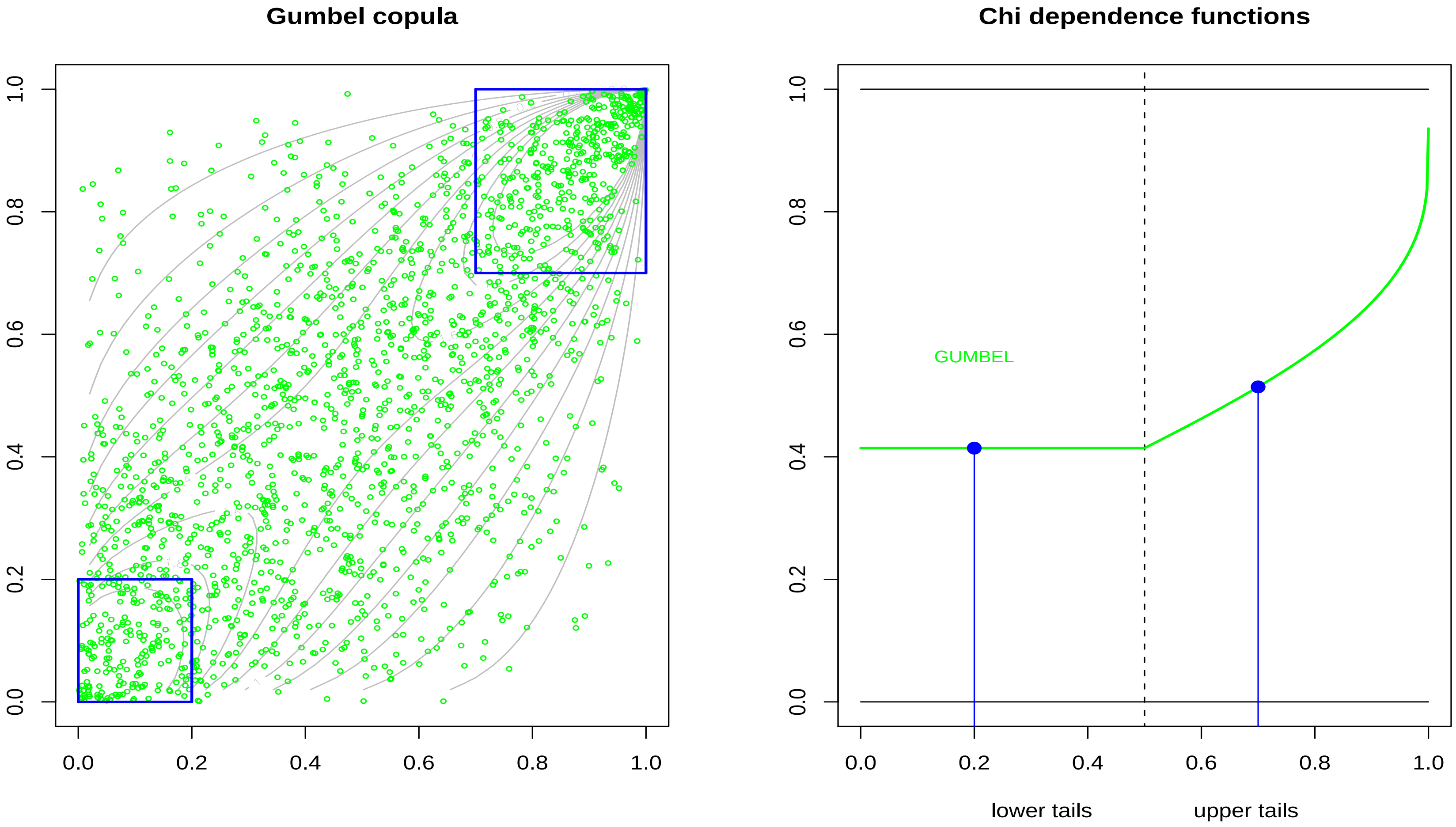

Fig. 19 – $\overline{\chi}$ functions.

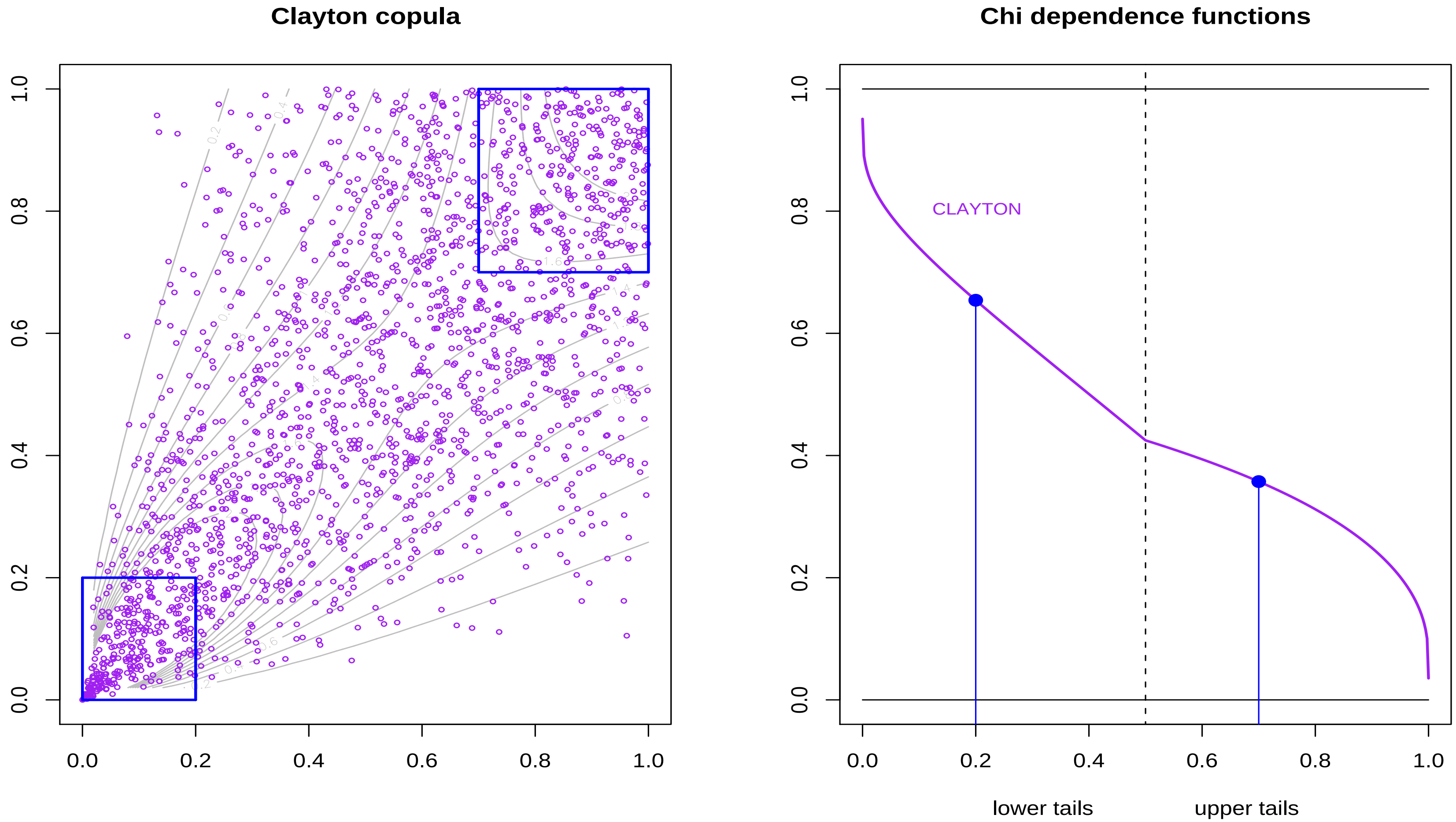

Fig. 20 – $\overline{\chi}$ functions.

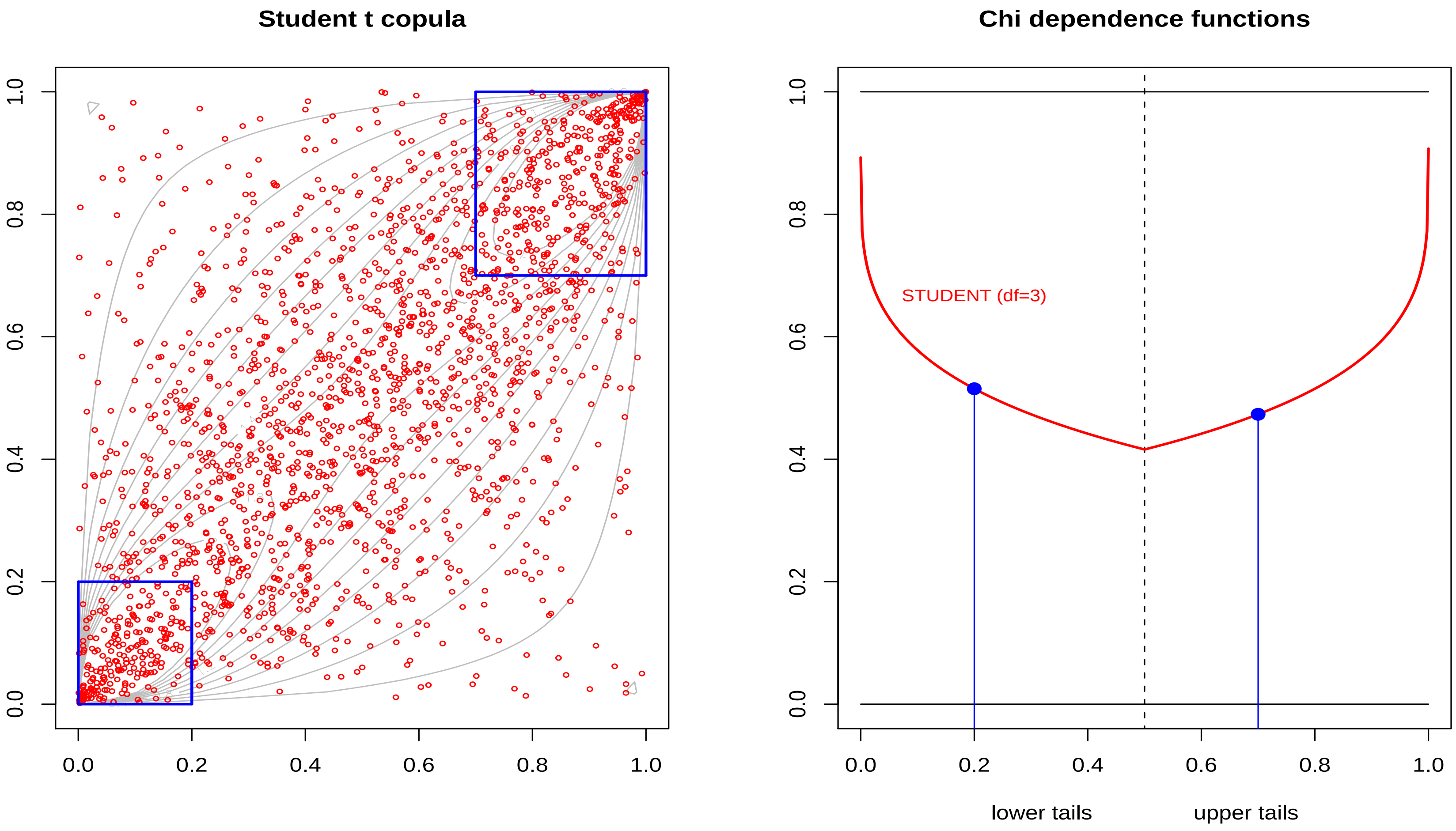

Fig. 21 – $\overline{\chi}$ functions.

Application in risk management : Loss-ALAE

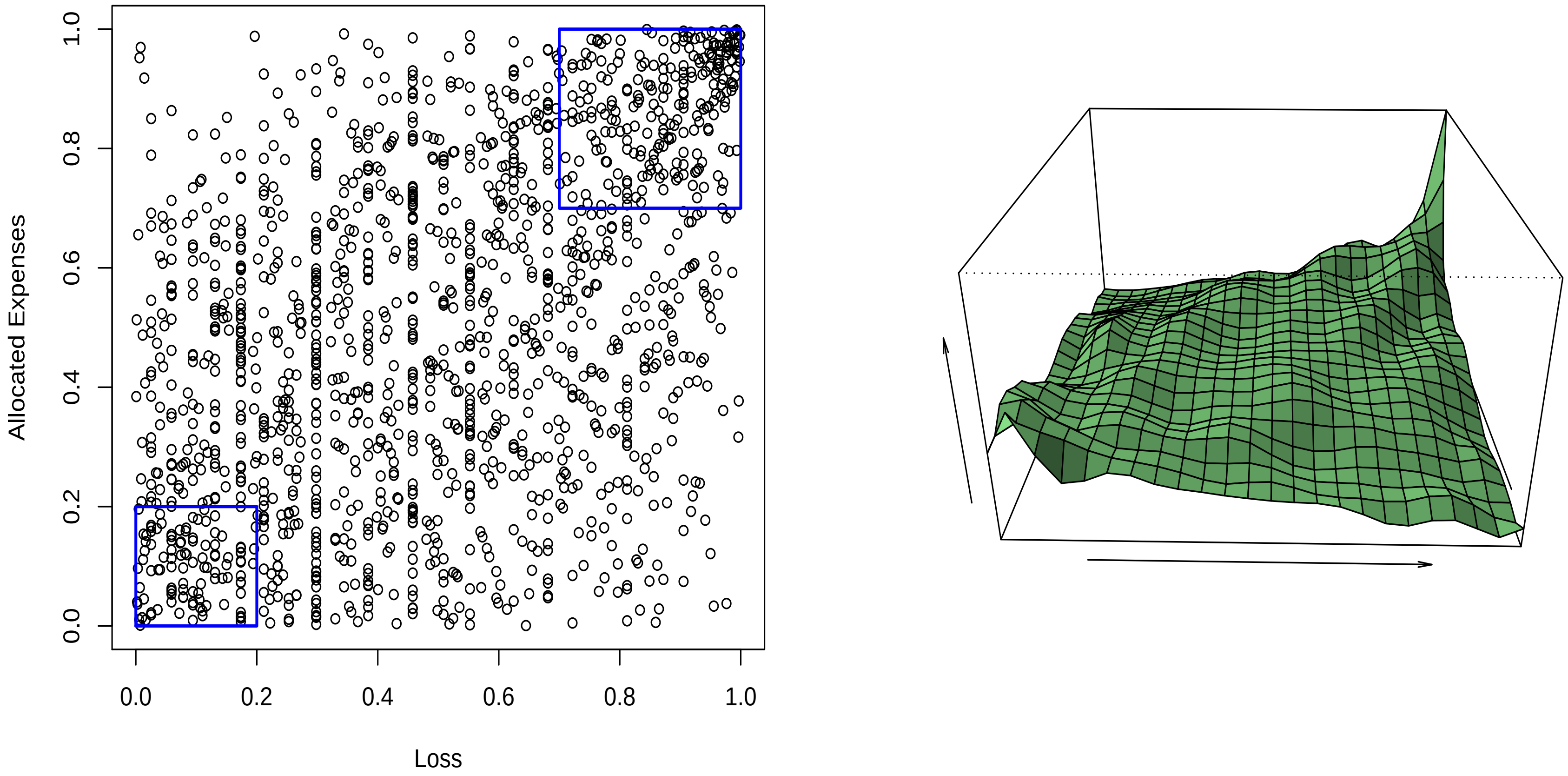

Fig. 22 – Losses and allocated expenses.

Application in risk management : Loss-ALAE

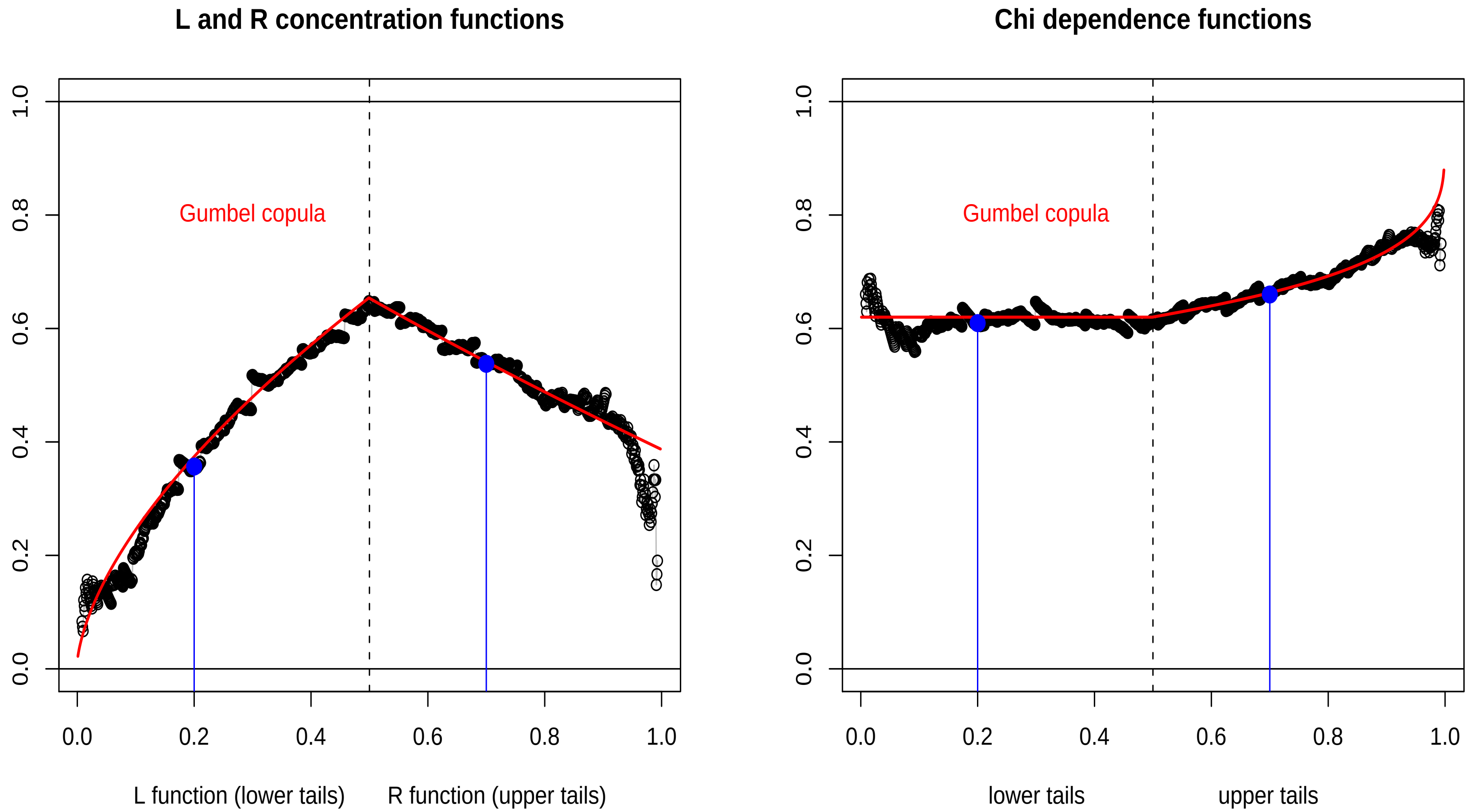

Fig. 23 – L and R cumulative curves, and $\overline{\chi}$ functions.

Application in risk management : car-household

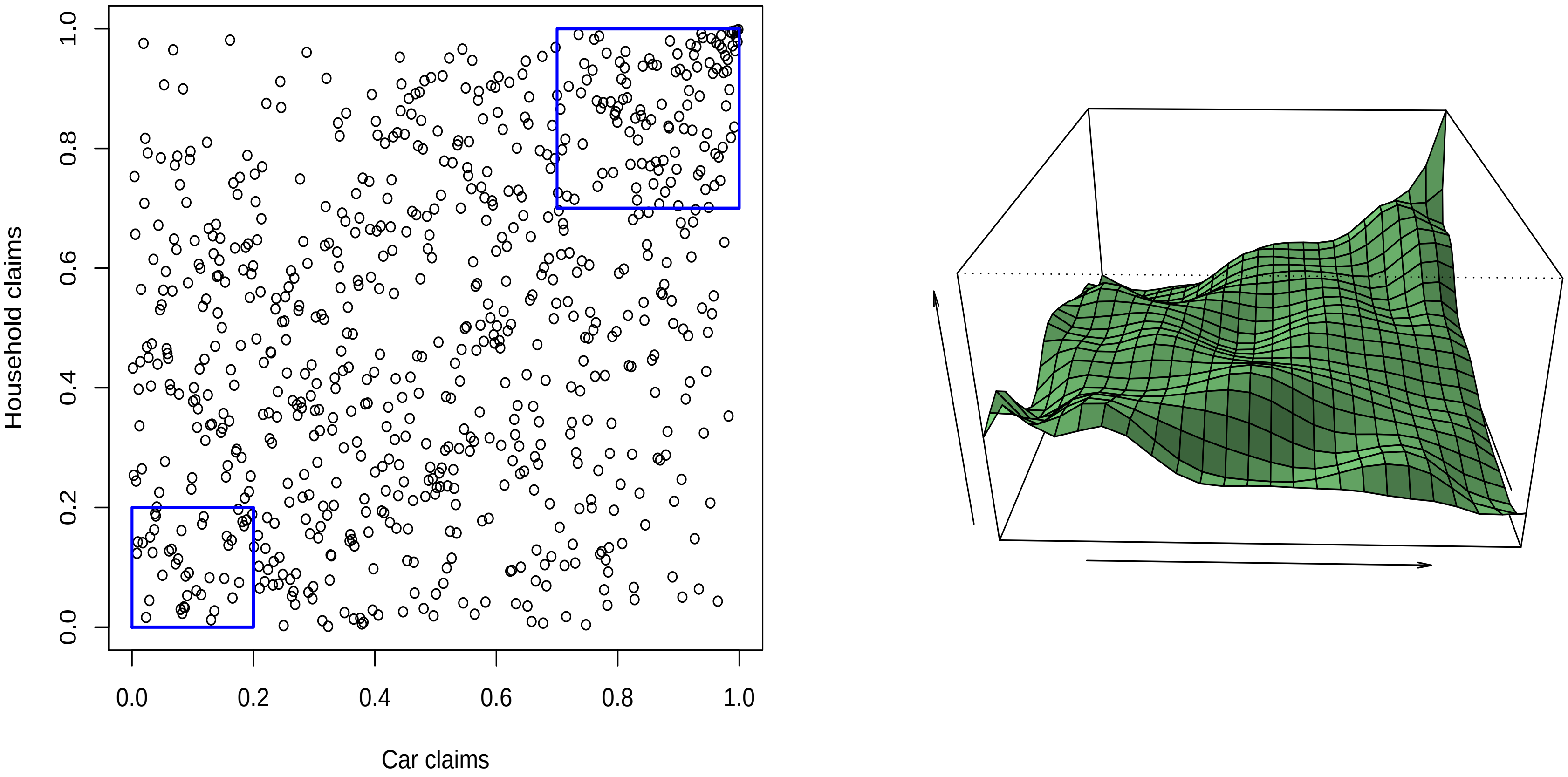

Fig. 24 – Motor and Household claims.

Application in risk management : car-household

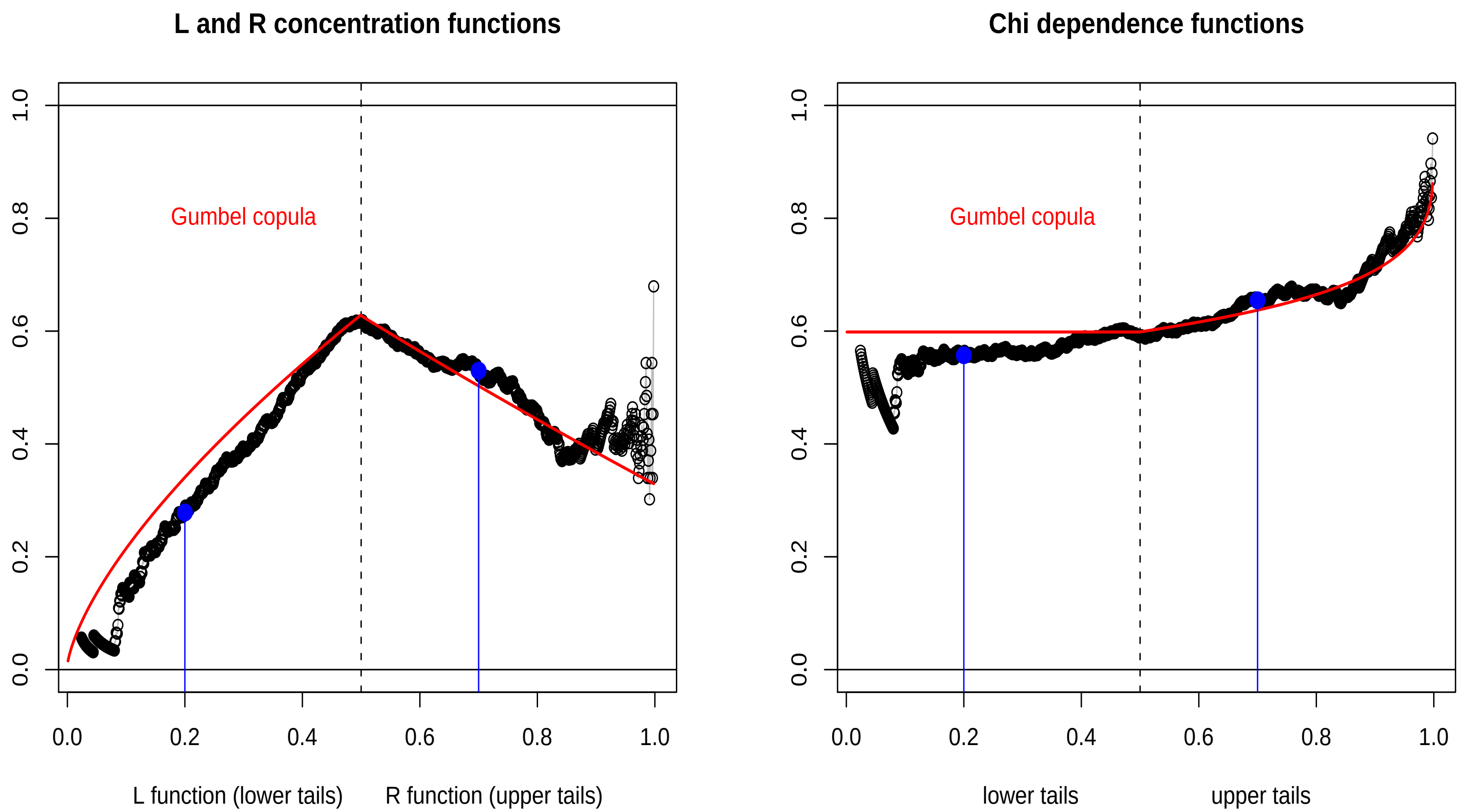

Fig. 25 – L and R cumulative curves, and $\overline{\chi}$ functions.

Case of Archimedean copulas

For an exhaustive study of tail behavior for Archimedean copulas, see Charpentier & Segers (2008).

- upper tail : function of $\phi'(1)$ and $\theta_1 = -\lim_{s\rightarrow 0} \dfrac{s\phi'(1-s)}{\phi(1-s)}$,

◦ $\phi'(1) < 0$: tail independence

◦ $\phi'(1) = 0$ and $\theta_1 = 1$: dependence in independence

◦ $\phi'(1) = 0$ and $\theta_1 > 1$: tail dependence

- lower tail : function of $\phi(0)$ and $\theta_0 = -\lim_{s\rightarrow 0} \dfrac{s\phi'(s)}{\phi(s)}$,

◦ $\phi(0) < \infty$: tail independence

◦ $\phi(0) = \infty$ and $\theta_0 = 0$: dependence in independence

◦ $\phi(0) = \infty$ and $\theta_0 > 0$: tail dependence

Measuring risks ?

the pure premium as a technical benchmark

Pascal, Fermat, Condorcet, Huygens, d'Alembert in the XVIIIth century proposed to evaluate the "*produit scalaire des probabilités et des gains*",

$$<\boldsymbol{p}, \boldsymbol{x}>=\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} \mathbb{P}(X = x_i) \cdot x_i = \mathbb{E}_{\mathbb{P}}(X),$$

based on the "*règle des parties*".

For Quételet, the expected value was, in the context of insurance, the price that guarantees a financial equilibrium.

From this idea, we consider in insurance the pure premium as $\mathbb{E}_{\mathbb{P}}(X)$. As in Cournot (1843), "*l'espérance mathématique est donc le juste prix des chances*" (or the "*fair price*" mentioned in Feller (1953)).

Problem : Saint Peterburg's paradox, i.e. infinite mean risks (cf. natural catastrophes)

the pure premium as a technical benchmark

For a positive random variable X, recall that $\mathbb{E}_{\mathbb{P}}(X) = \displaystyle\int_0^\infty \mathbb{P}(X > x)dx$.

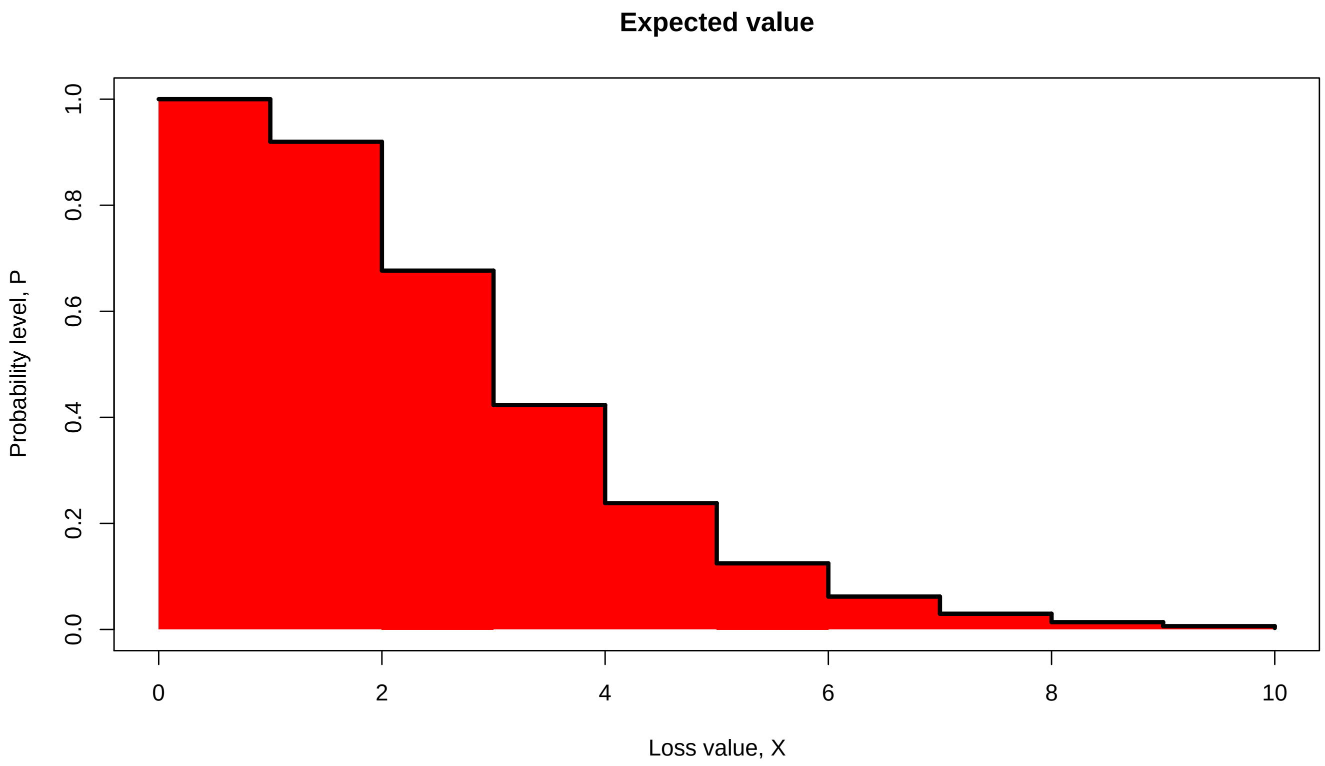

Fig. 26 – Expected value $\mathbb{E}_{\mathbb{P}}(X) = \int x dF_X(x) = \int \mathbb{P}(X > x)dx$.

from pure premium to expected utility principle

$$\mathcal{R}_u(X) = \int u(x)d\mathbb{P} = \int \mathbb{P}(u(X) > x))dx$$

where $u : [0, \infty) \to [0, \infty)$ is a utility function.

Example with an exponential utility, $u(x) = [1 - e^{-\alpha x}]/\alpha$,

$$\mathcal{R}_u(X) = \frac{1}{\alpha} \log \left(\mathbb{E}_{\mathbb{P}}(e^{\alpha X}) \right),$$

i.e. the entropic risk measure.

See Cramer (1728), Bernoulli (1738), von Neumann & Morgenstern (1944), Rochet (1994)... etc.

Distortion of values versus distortion of probabilities

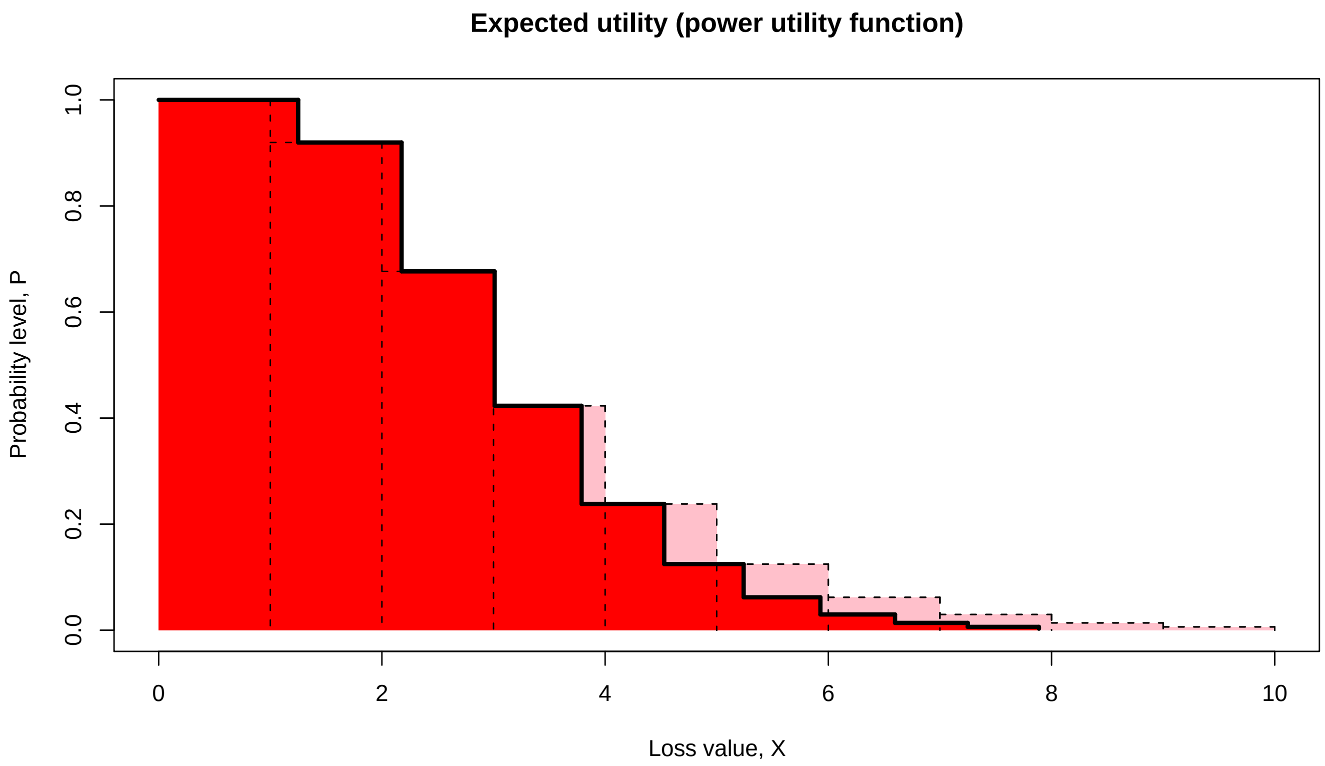

Fig. 27 – Expected utility $\int u(x) dF_X(x)$.

Distortion of values versus distortion of probabilities

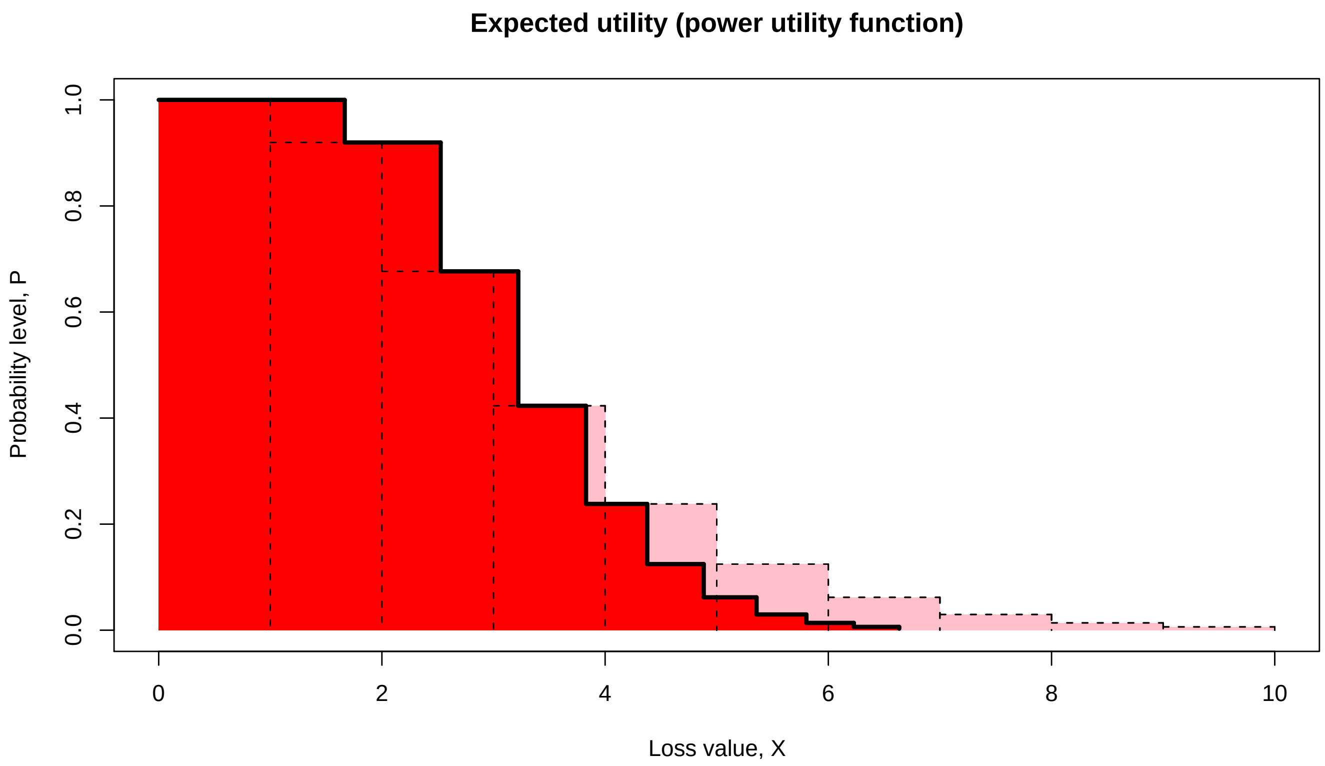

FIG. 28 – Expected utility $\int u(x) dF_X(x)$.

from pure premium to distorted premiums (Wang)

$$\mathcal{R}_g(X) = \int x dg \circ \mathbb{P} = \int g(\mathbb{P}(X > x))dx$$

where $g : [0,1] \to [0,1]$ is a distorted function.

Example

- if $g(x) = \mathbb{I}(X \geq 1 - \alpha)$ $\mathcal{R}_g(X) = VaR(X, \alpha)$,
- if $g(x) = \min\{x/(1-\alpha), 1\}$ $\mathcal{R}_g(X) = TVaR(X, \alpha)$ (also called expected shortfall), $\mathcal{R}_g(X) = \mathbb{E}_{\mathbb{P}}(X|X > VaR(X, \alpha))$.

See D'Alembert (1754), Schmeidler (1986, 1989), Yaari (1987), Denneberg (1994)... etc.

Remark : $\mathcal{R}_g(X)$ might be denoted $\mathbb{E}_{g\circ\mathbb{P}}$. But it is not an expected value since $\mathbb{Q} = g \circ \mathbb{P}$ is *not* a probability measure.

Distortion of values versus distortion of probabilities

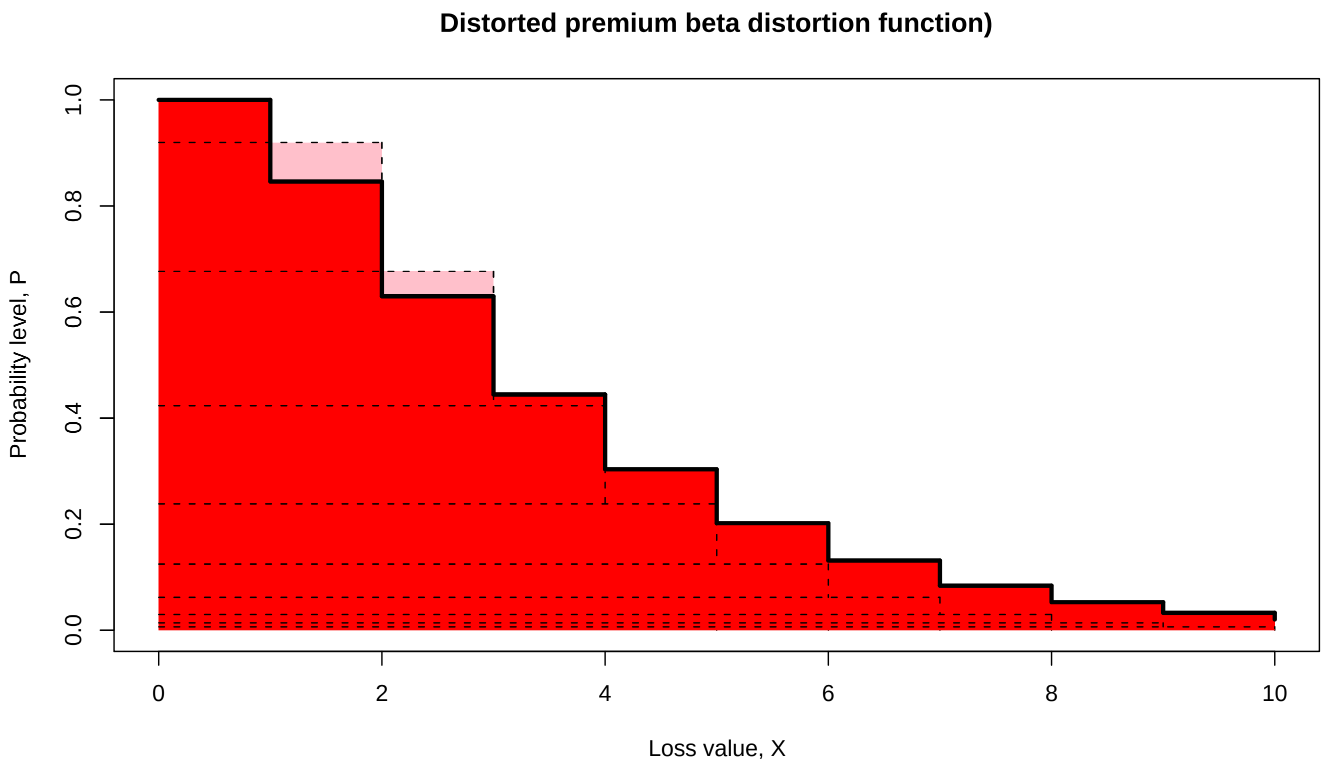

Fig. 29 – Distorted probabilities $\int g(\mathbb{P}(X > x))dx$.

Distortion of values versus distortion of probabilities

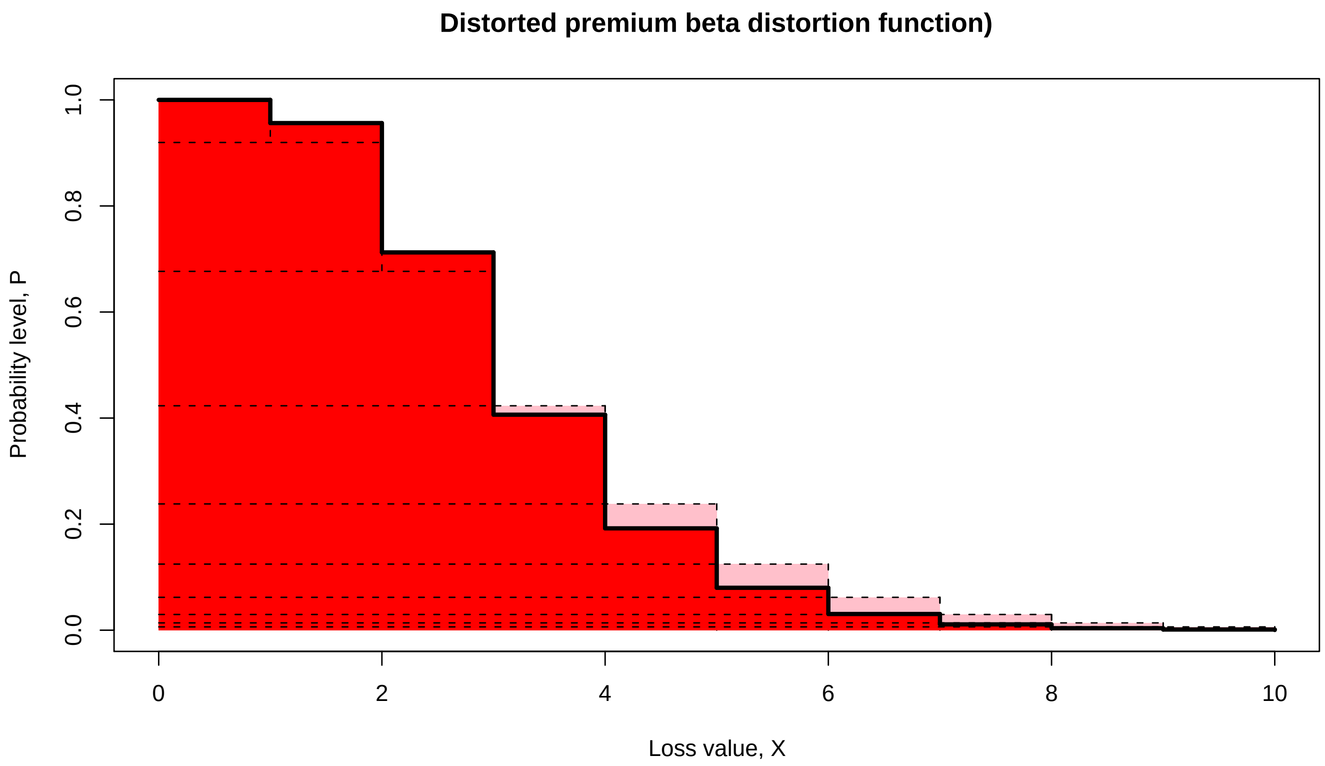

Fig. 30 – Distorted probabilities $\int g(\mathbb{P}(X > x))dx$.

some particular cases a *classical* premiums

The exponential premium or **entropy measure** : obtained when the agent as an exponential utility function, i.e.

$$\pi \text{ such that } U(\omega-\pi)=\mathbb{E}_{\mathbb{P}}(U(\omega-S)), U(x)=-\exp(-\alpha x),$$

i.e. $\pi=\dfrac{1}{\alpha}\log\mathbb{E}_{\mathbb{P}}(e^{\alpha X})$.

Esscher's transform (see Esscher (1936), Bühlmann (1980)),

$$\pi=\mathbb{E}_{\mathbb{Q}}(X)=\frac{\mathbb{E}_{\mathbb{P}}(X\cdot e^{\alpha X})}{\mathbb{E}_{\mathbb{P}}(e^{\alpha X})},$$

for some $\alpha>0$, i.e.

$$\frac{d\mathbb{Q}}{d\mathbb{P}}=\frac{e^{\alpha X}}{\mathbb{E}_{\mathbb{P}}(e^{\alpha X})}.$$

Wang's premium (see Wang (2000)), extending the Sharp ratio concept

$$\mathbb{E}(X)=\int_0^\infty \overline{F}(x)dx \text{ and } \pi=\int_0^\infty \Phi(\Phi^{-1}(\overline{F}(x))+\lambda)dx$$

Risk measures

The two most commonly used risk measures for a random variable X (assuming that a loss is positive) are, $q \in (0,1)$,

- Value-at-Risk (VaR),

$$VaR_q(X) = \inf\{x \in \mathbb{R}, \mathbb{P}(X > x) \leq \alpha\},$$

- Expected Shortfall (ES), Tail Conditional Expectation (TCE) or Tail Value-at-Risk (TVaR)

$$TVaR_q(X) = \mathbb{E}\left(X|X > VaR_q(X)\right),$$

Artzner, Delbaen, Eber & Heath (1999) : a *good* risk measure is subadditive,

TVaR is subadditive, VaR is not subadditive (in general).

Risk measures and diversification

Any copula C is bounded by Frchet-Hoeffding bounds,

$$\max\left\{\sum_{i=1}^{d} u_i - (d-1), 0\right\} \leq C(u_1,\dots,u_d) \leq \min\{u_1,\dots,u_d\},$$

and thus, any distribution F on $\mathcal{F}(F_1,\dots,F_d)$ is bounded

$$\max\left\{\sum_{i=1}^{d} F_i(x_i) - (d-1), 0\right\} \leq F(x_1,\dots,x_d) \leq \min\{F_1(x_1),\dots,F_f(x_d)\}.$$

Does this means the comonotonicity is always the worst-case scenario ?

Given a random pair (X, Y), let (X^-, Y^-) and (X^+, Y^+) denote contercomonotonic and comonotonic versions of (X, Y), do we have

$$\mathcal{R}(\phi(X^-, Y^-)) \overset{?}{\leq} \mathcal{R}(\phi(X^, Y^)) \overset{?}{\leq} \mathcal{R}(\phi(X^+, Y^+)).$$

Tchen's theorem and bounding some pure premiums

If $\phi : \mathbb{R}^2 \to \mathbb{R}$ is supermodular, i.e.

$$\phi(x_2, y_2) - \phi(x_1, y_2) - \phi(x_2, y_1) + \phi(x_1, y_1) \geq 0,$$

for any $x_1 \leq x_2$ and $y_1 \leq y_2$, then if $(X, Y) \in \mathcal{F}(F_X, F_Y)$,

$$\mathbb{E}\left(\phi(X^-, Y^-)\right) \leq \mathbb{E}\left(\phi(X, Y)\right) \leq \mathbb{E}\left(\phi(X^+, Y^+)\right),$$

as proved in Tchen (1981).

Example 7. *the stop loss premium for the sum of two risks* $\mathbb{E}((X + Y - d)_+)$ *is supermodular.*

Example 8. *For the n-year joint-life annuity,*

$$a_{xy:\overline{n}|} = \sum_{k=1}^{n} v^k \mathbb{P}(T_x > k \text{ and } T_y > k) = \sum_{k=1}^{n} v^k {}_k p_{xy}.$$

Then

$$a^-_{xy:\overline{n}|} \leq a_{xy:\overline{n}|} \leq a^+_{xy:\overline{n}|},$$

where

$$a^-_{xy:\overline{n}|} = \sum_{k=1}^{n} v^k \max\{{}_k p_x + {}_k p_y - 1, 0\} (\text{ lower Frchet bound }),$$

$$a^+_{xy:\overline{n}|} = \sum_{k=1}^{n} v^k \min\{{}_k p_x, {}_k p_y\} (\text{ upper Frchet bound }).$$

Makarov's theorem and bounding Value-at-Risk

In the case where $\mathcal{R}$ denotes the Value-at-Risk (i.e. quantile function of the P&L distribution),

$$\mathcal{R}^- \leq \mathcal{R}(X^- + Y^-) \not\leq \mathcal{R}(X+Y) \not\leq \mathcal{R}(X^+ + Y^+) \leq \mathcal{R}^+,$$

where e.g. $\mathcal{R}^+$ can exceed the comonotonic case. Recall that

$$\mathcal{R}(X+Y) = \text{VaR}_q[X+Y] = F_{X+Y}^{-1}(q) = \inf\{x \in \mathbb{R} | F_{X+Y}(x) \geq q\}.$$

Proposition 9. *Let $(X,Y) \in \mathcal{F}(F_X, F_Y)$ then for all $s \in \mathbb{R}$,*

$$\tau_{C^-}(F_X, F_Y)(s) \leq \mathbb{P}(X+Y \leq s) \leq \rho_{C^-}(F_X, F_Y)(s),$$

where

$$\tau_C(F_X, F_Y)(s) = \sup_{x,y \in \mathbb{R}} \{C(F_X(x), F_Y(y)), x+y=s\}$$

and, if $\tilde{C}(u,v) = u+v-C(u,v)$,

$$\rho_C(F_X, F_Y)(s) = \inf_{x,y \in \mathbb{R}} \{\tilde{C}(F_X(x), F_Y(y)), x+y=s\}.$$

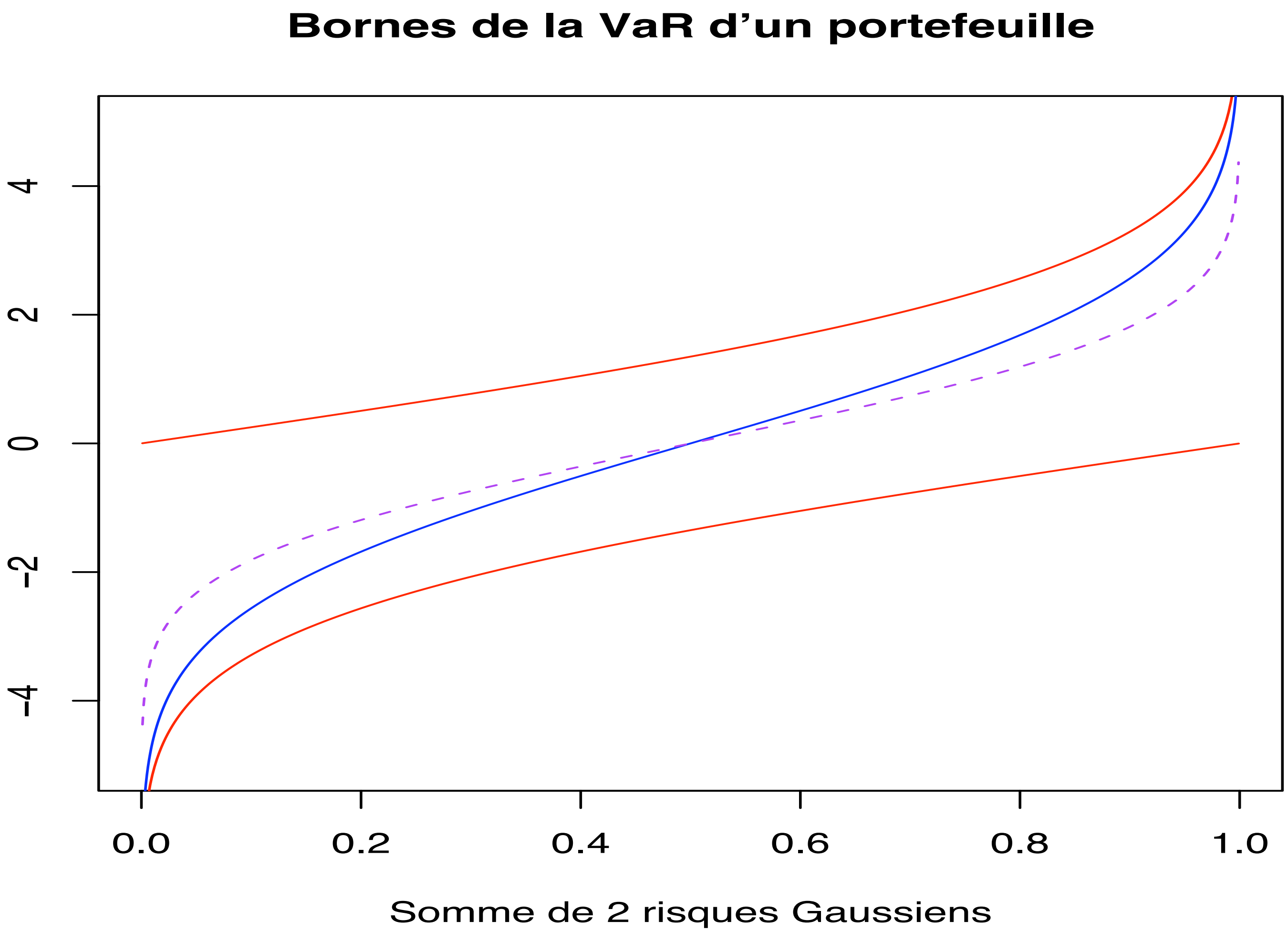

Fig. 31 – Value-at-Risk for 2 Gaussian risks $\mathcal{N}(0,1)$.

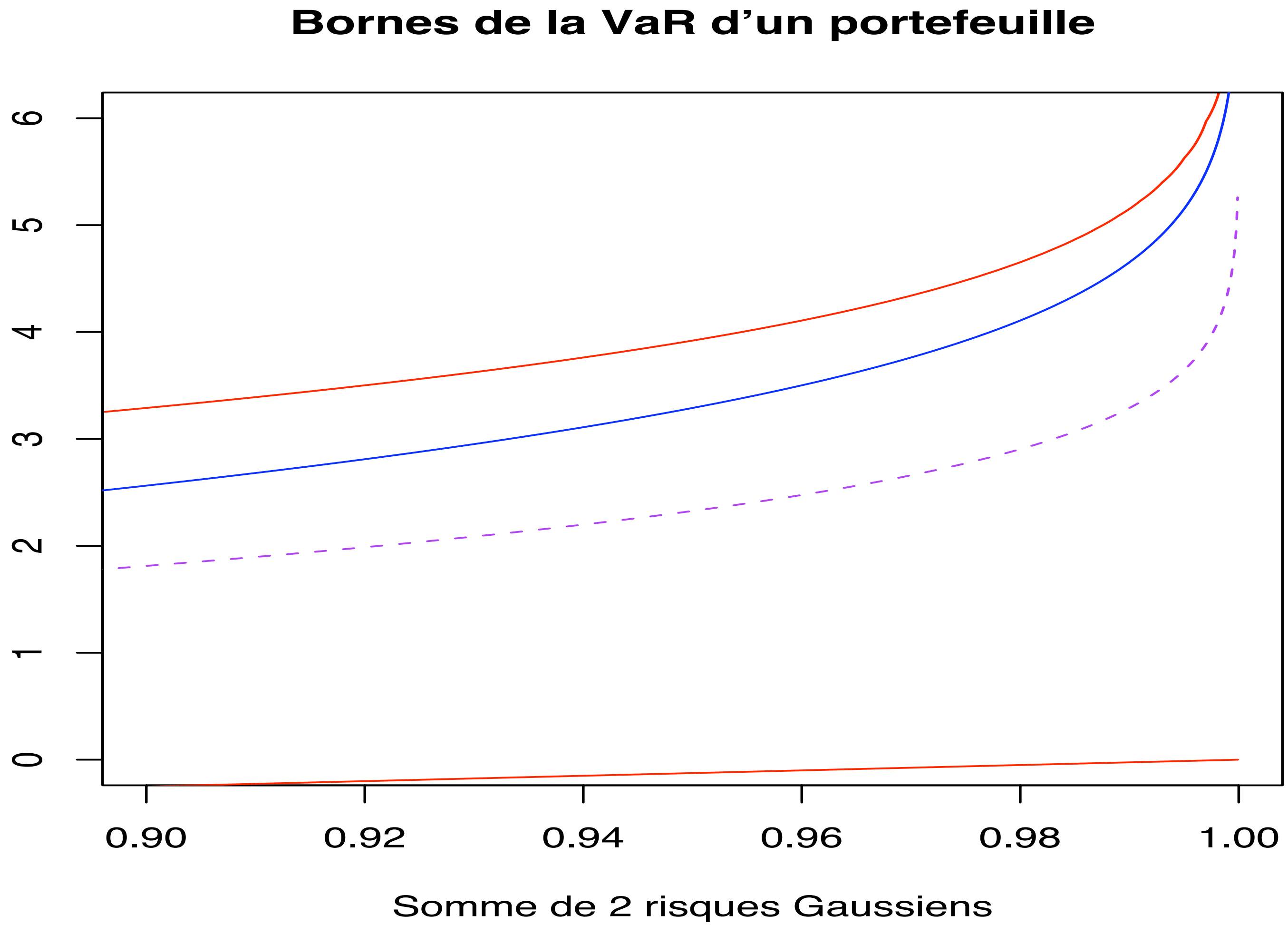

Fig. 32 – Value-at-Risk for 2 Gaussian risks $\mathcal{N}(0,1)$.

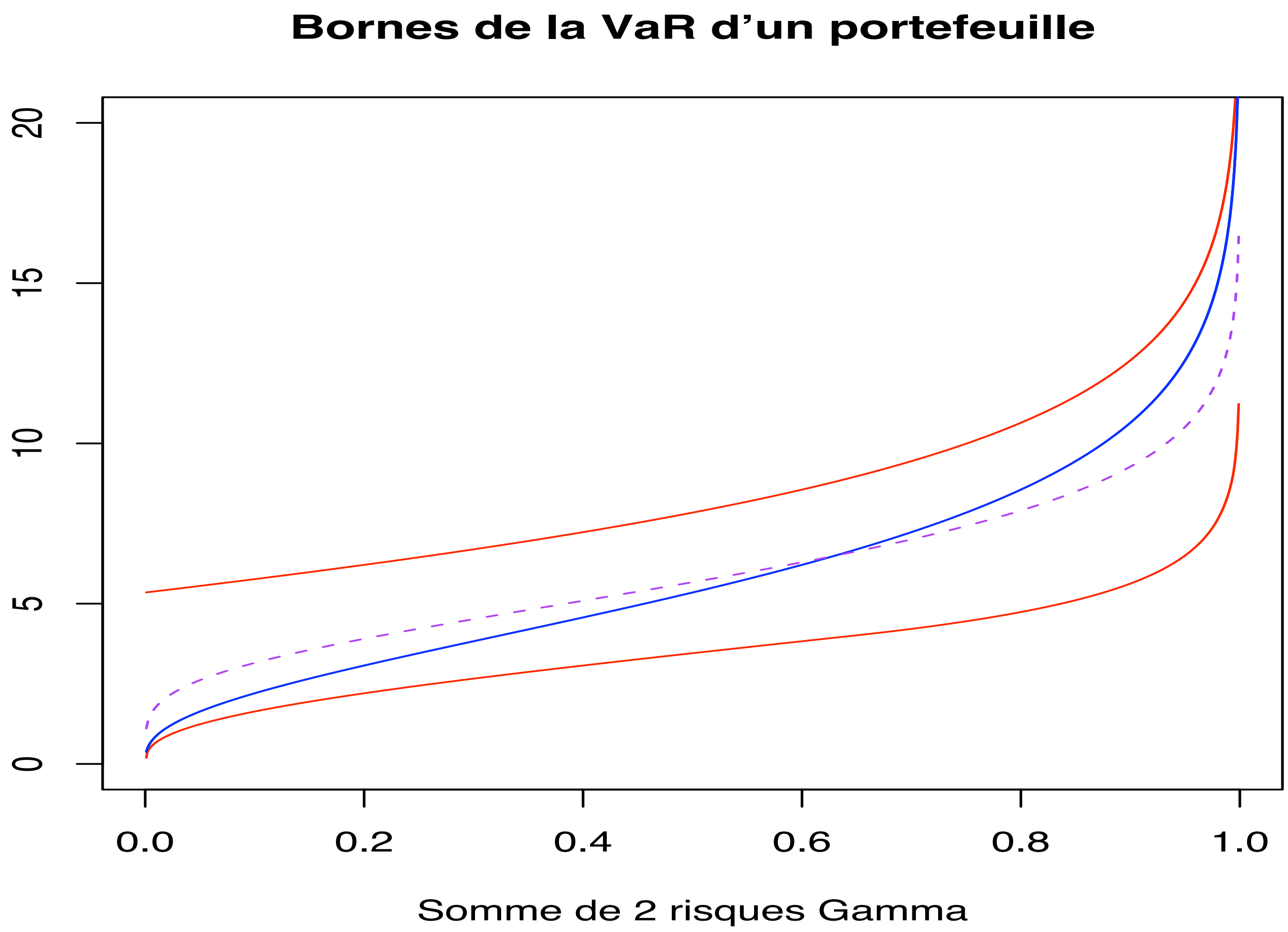

Fig. 33 – Value-at-Risk for 2 Gamma risks $\mathcal{G}(3, 1)$.

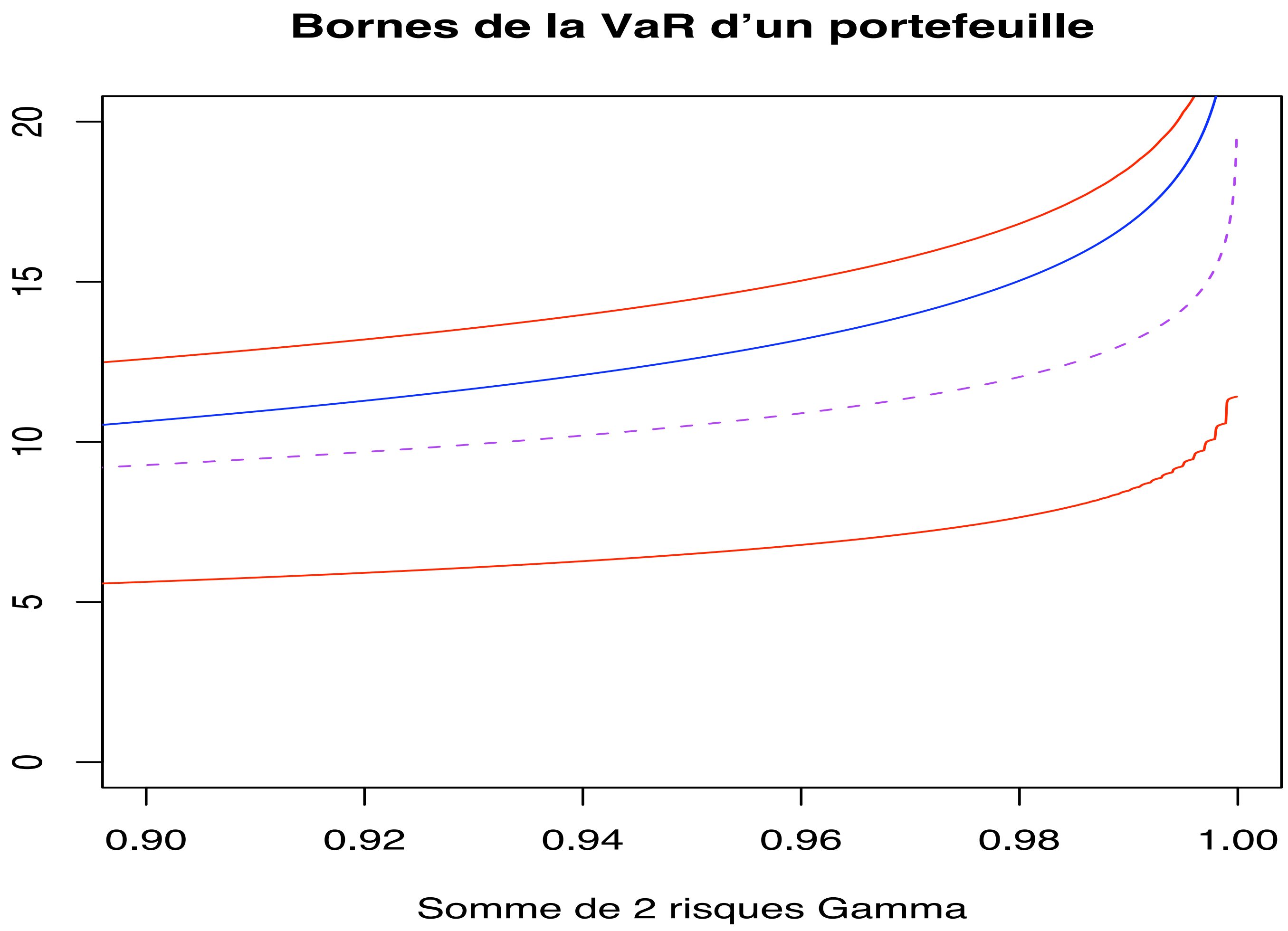

Fig. 34 – Value-at-Risk for 2 Gamma risks $\mathcal{G}(3,1)$.

The more correlated, the more risky ?

Will the risk of the portfolio increase with correlation ?

Recall the following theoretical result :

Proposition 10. *Assume that $\boldsymbol{X}$ and $\boldsymbol{X}'$ are in the same Fréchet space (i.e. $X_i \stackrel{\mathcal{L}}{=} X_i'$), and define*

$$S = X_1 + \cdots + X_n \text{ and } S' = X_1' + \cdots + X_n'.$$

If $\boldsymbol{X} \preceq \boldsymbol{X}'$ for the concordance order, then $S \preceq_{TVaR} S'$ for the stop-loss *or* TVaR *order.*

A consequence is that if $\boldsymbol{X}$ and $\boldsymbol{X}'$ are exchangeable,

$$corr(X_i, X_j) \leq corr(X_i', X_j') \implies TVaR(S, p) \leq TVaR(S', p), \text{ for all } p \in (0, 1).$$

See Müller & Stoyen (2002) for some possible extensions.

The more correlated, the more risky ?

Consider

- d lines of business,
- simply a binomial distribution on each line of business, with small loss probability (e.g. $\pi = 1/1000$).

Let $\begin{cases} 1 \text{ if there is a claim on line } i \\ 0 \text{ if not} \end{cases}$, and $S = X_1 + \cdots + X_d$.

Will the correlation among the X_i's increase the Value-at-Risk of S ?

Consider a probit model, i.e. $X_i = \mathbf{1}(X_i^\star \leq u_i)$, where $\boldsymbol{X}^\star \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$, i.e. a Gaussian copula.

Assume that $\mathbf{\Sigma} = [\sigma_{i,j}]$ where $\sigma_{i,j} = \rho \in [-1, 1]$ when $i \neq j$.

The more correlated, the more risky ?

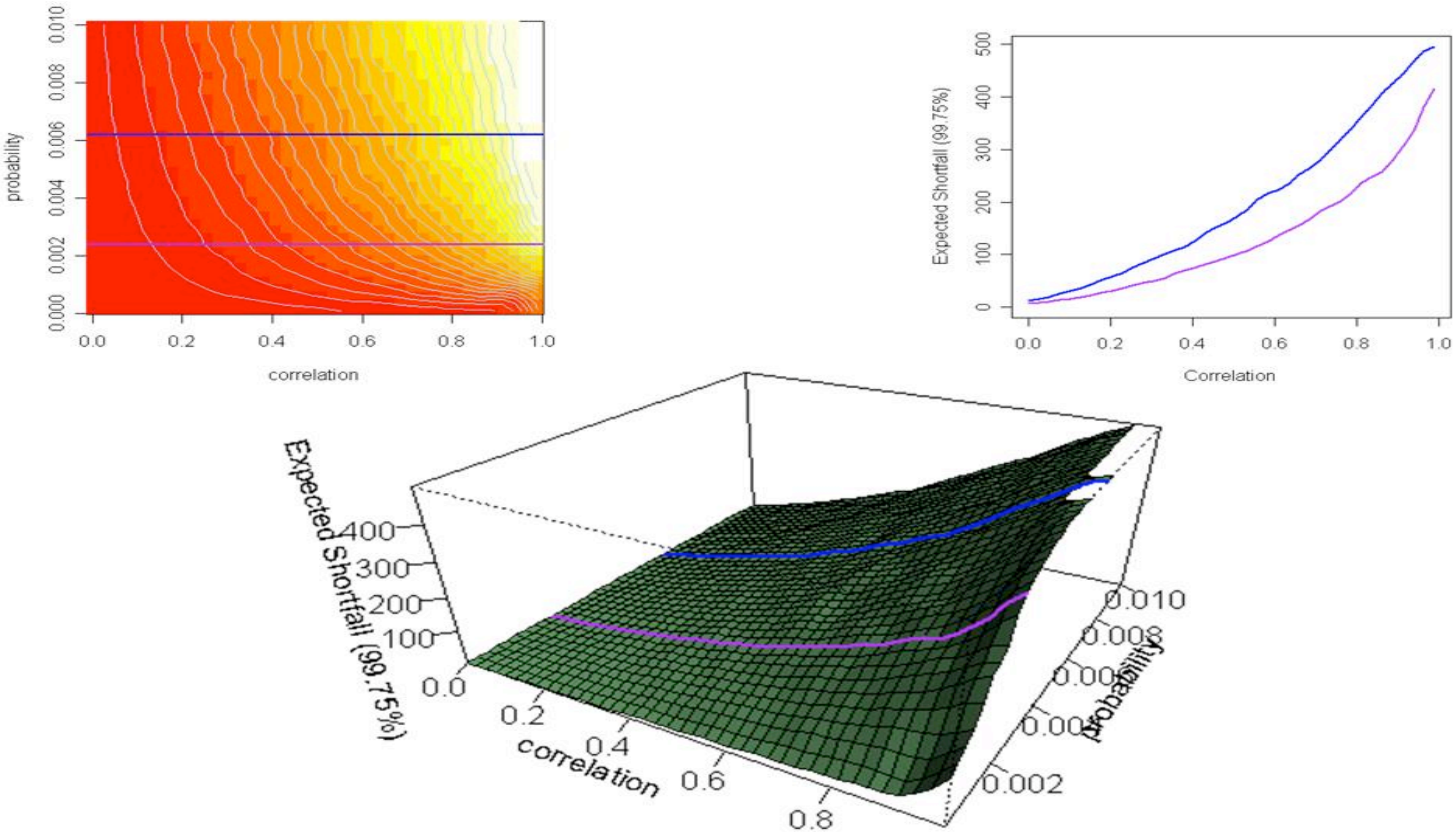

Fig. 35 – 99.75% TVaR (or expected shortfall) for Gaussian copulas.

The more correlated, the more risky ?

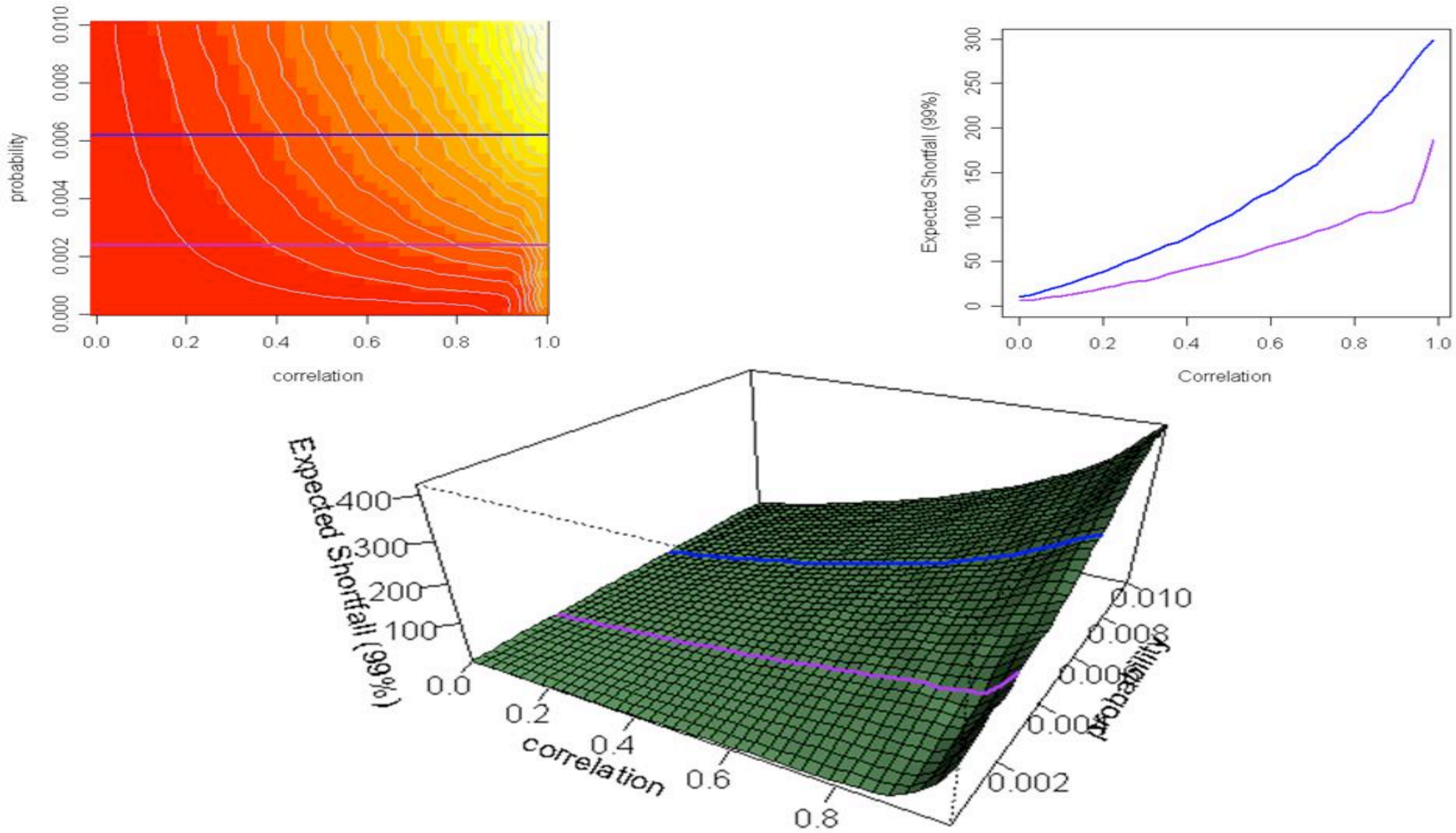

Fig. 36 – 99% TVaR (or expected shortfall) for Gaussian copulas.

The more correlated, the more risky ?

What about other risk measures, e.g. Value-at-Risk ?

$$corr(X_i, X_j) \leq corr(X_i', X_j') \not\Rightarrow VaR(S,p) \leq VaR(S',p), \text{ for all } p \in (0,1).$$

(see e.g. Mittnik & Yener (2008)).

The more correlated, the more risky ?

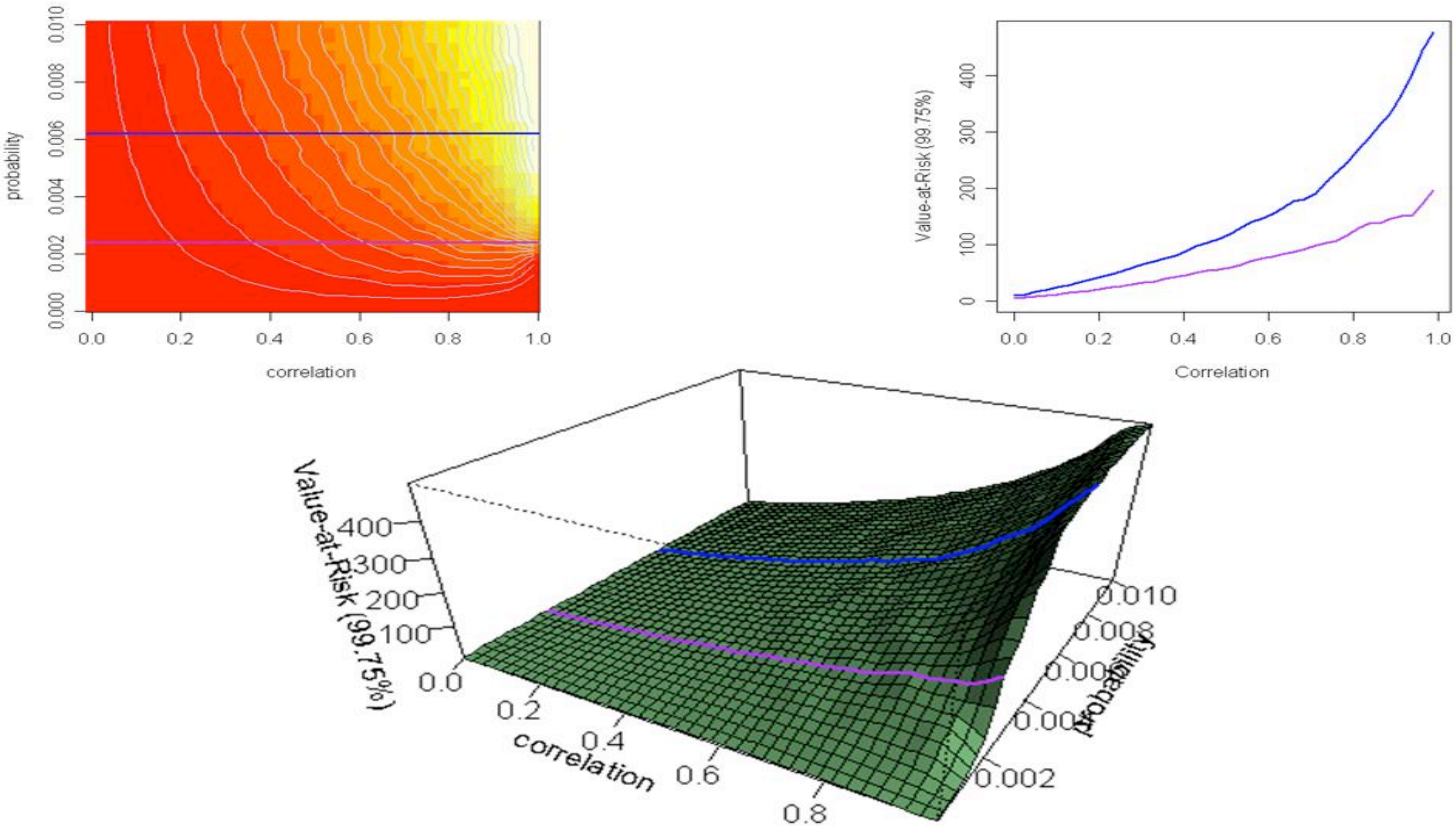

Fig. 37 – 99.75% VaR for Gaussian copulas.

The more correlated, the more risky ?

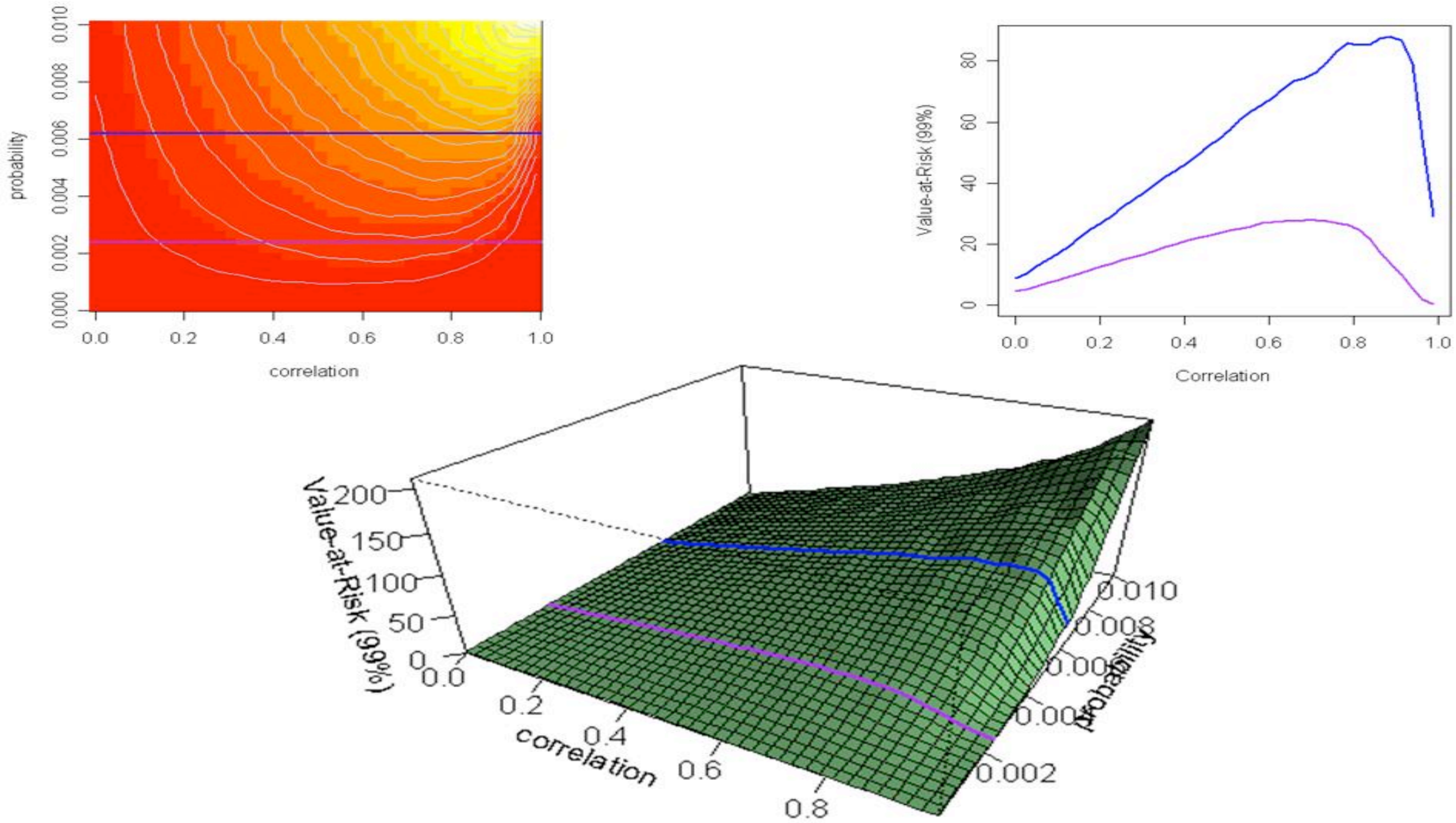

Fig. 38 – 99% VaR for Gaussian copulas.

The more correlated, the more risky ?

What could be the impact of tail dependence ?

Previously, we considered a Gaussian copula, i.e. tail independence. What if there was tail dependence ?

Consider the case of a Student t-copula, with ν degrees of freedom.

The more correlated, the more risky ?

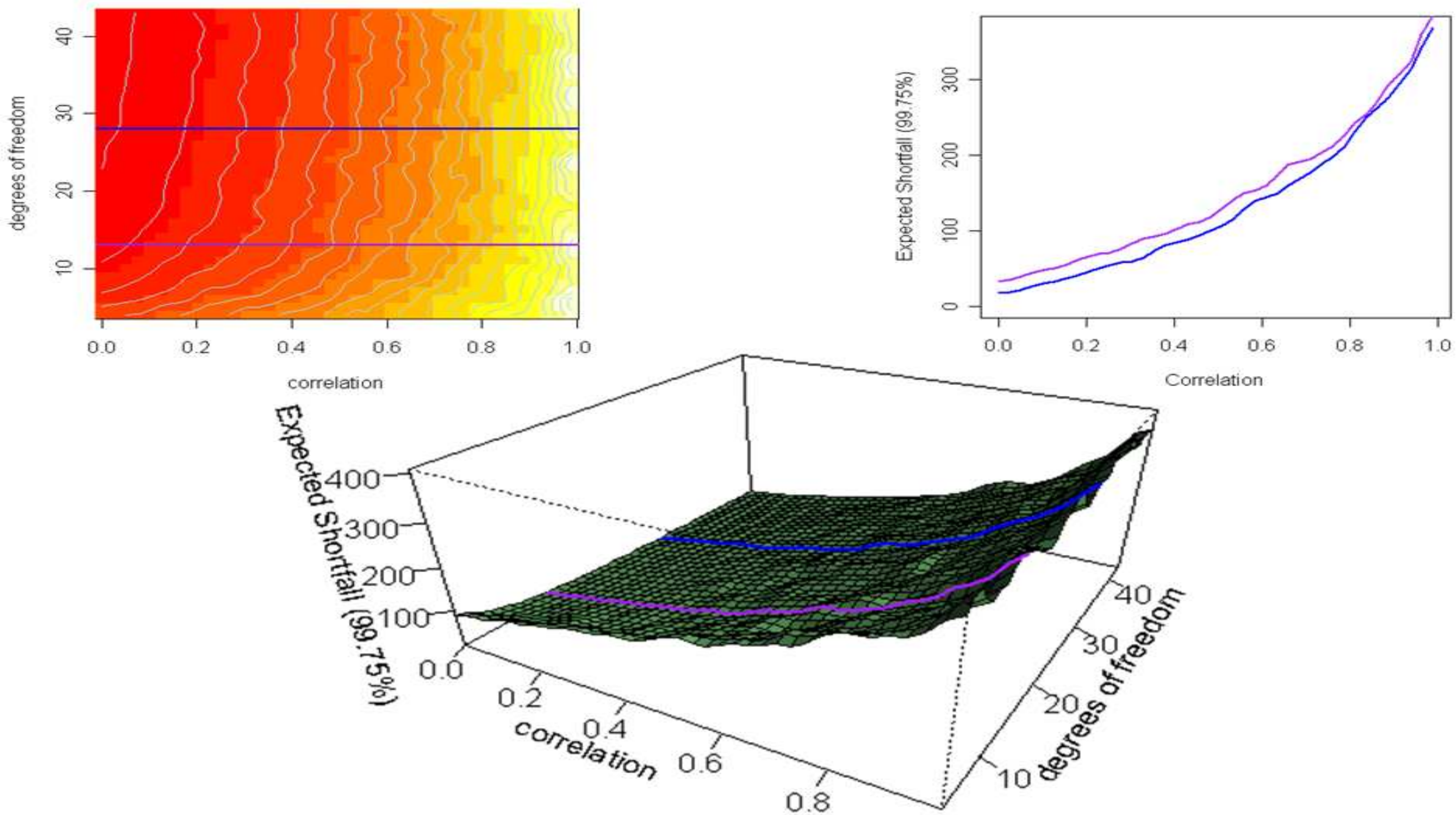

Fig. 39 – 99.75% TVaR (or expected shortfall) for Student t-copulas.

The more correlated, the more risky ?

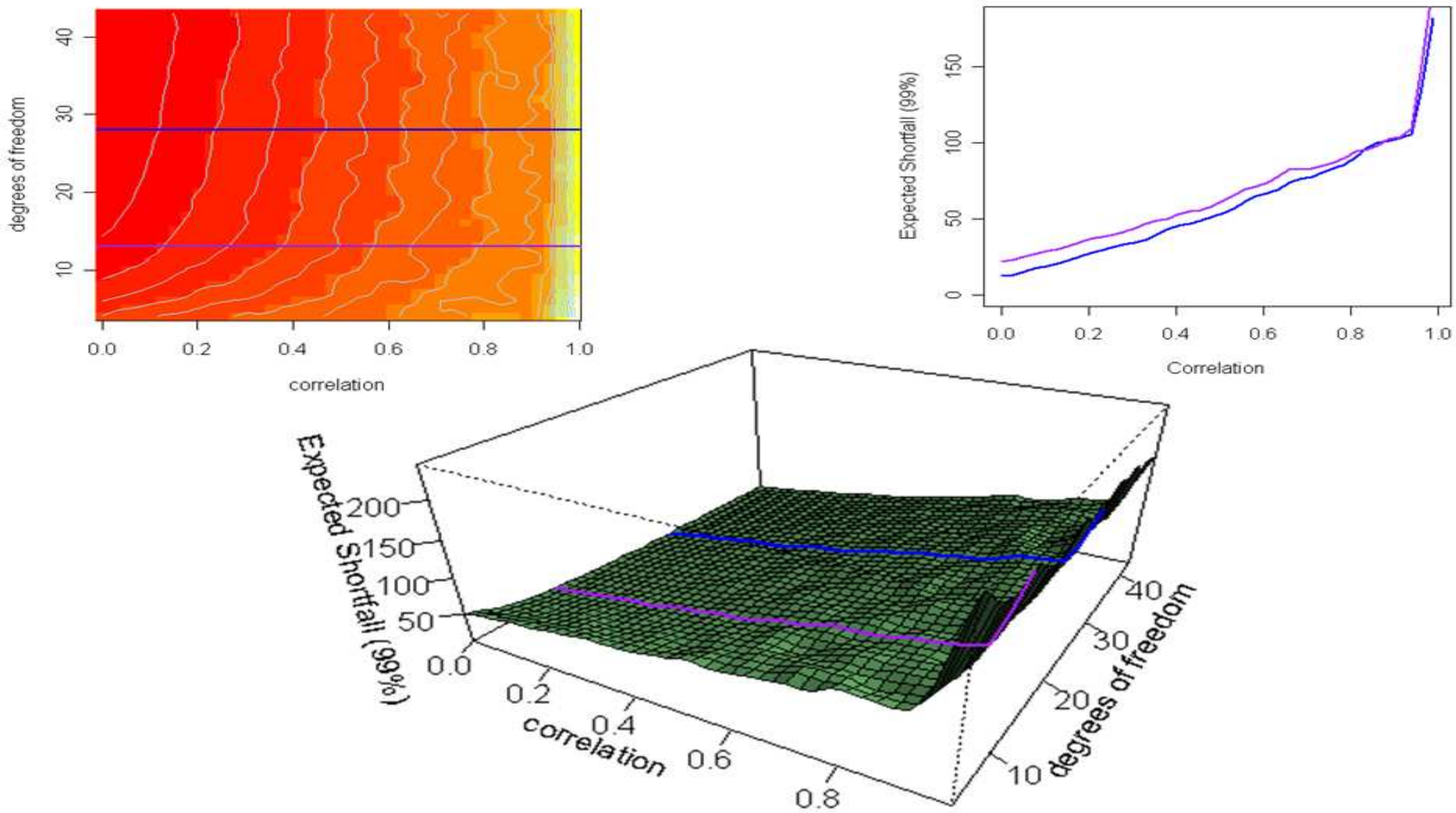

Fig. 40 – 99% TVaR (or expected shortfall) for Student t-copulas.

The more correlated, the more risky ?

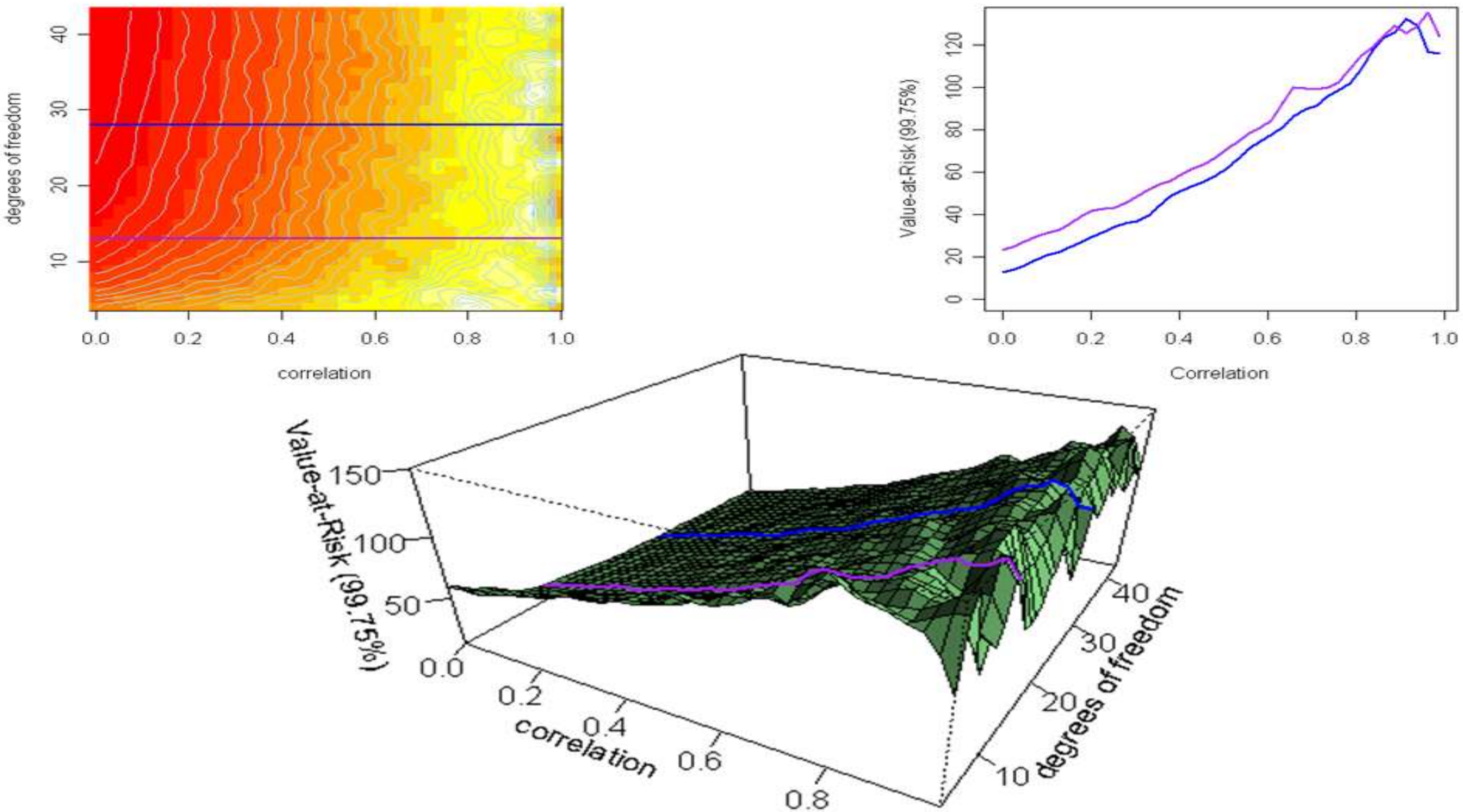

Fig. 41 – 99.75% VaR for Student t-copulas.

The more correlated, the more risky ?

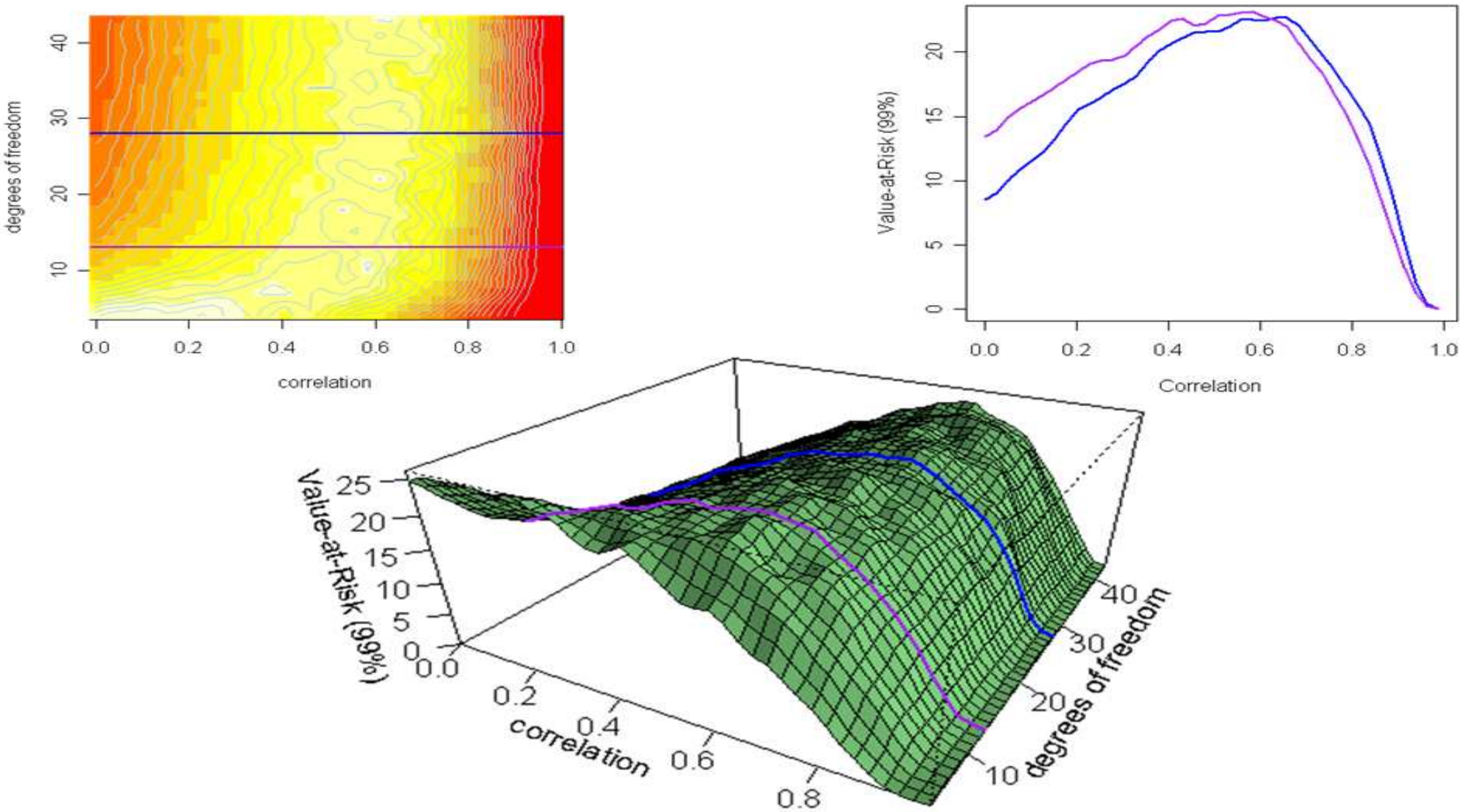

Fig. 42 – 99% VaR for Student t-copulas.

Conclusion

- (standard) correlation is definitively not an appropriate tool to describe dependence features,
  - in order to fully describe dependence, use copulas,
  - since major focus in risk management is related to extremal event, focus on tail dependence meausres,
- which copula can be appropriate ?
  - Elliptical copulas offer a nice and simple parametrization, based on pairwise comparison,
  - Archimedean copulas might be too restrictive, but possible to introduce Hierarchical Archimedean copulas,
- Value-at-Risk might yield to non-intuitive results,
  - need to get a better understanding about Value-at-Risk pitfalls,
  - need to consider alternative downside risk measures (namely TVaR).