Quantile and Expectile Regressions

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Erasmus School of Economics, May 2017

http://freakonometrics.hypotheses.org

Econometrics vs Machine Learning

As claimed in Freedman (2005), Statistical Models, in **econometrics**, given some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, assume that y_i are realization of i.i.d. variables Y_i (given $\boldsymbol{X}_i = \boldsymbol{x}_i$) with distribution F_{m_i} ("*conditional distribution story*" or "*causal story*"). E.g.

$$Y|\boldsymbol{X} = \boldsymbol{x} \sim \mathcal{N}(\underbrace{\boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}}_{m(\boldsymbol{x})}, \sigma^2) \text{ or } Y|\boldsymbol{X} = \boldsymbol{x} \sim \mathcal{B}(m(\boldsymbol{x})) \text{ where } m(\boldsymbol{x}) = \frac{e^{\boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}}}{1 + e^{\boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}}}$$

Then solve ("*maximum likelihood*" framework)

$$\widehat{m}(\cdot) = \underset{m(\cdot)\in\mathcal{F}}{\text{argmax}}\left\{ \log \mathcal{L}(m(\boldsymbol{x}); \boldsymbol{y})\right\} = \underset{m(\cdot)\in\mathcal{F}}{\text{argmax}}\left\{\sum_{i=1}^n \log f_{m(\boldsymbol{x}_i)}(y_i)\right\}$$

where $\log \mathcal{L}$ denotes the log-likelihood, see Haavelmo (1944) The Probability Approach in Econometrics.

Econometrics vs Machine Learning

In **machine learning** ("*explanatory data story*" in Freedman (2005), Statistical Models), given some dataset $(\boldsymbol{x}_i, y_i)$, solve

$$m^\star(\cdot) = \underset{m(\cdot)\in\mathcal{F}}{\text{argmin}} \left\{ \sum_{i=1}^{n} \ell(m(\boldsymbol{x}_i), y_i) \right\}$$

for some loss functions $\ell(\cdot,\cdot)$. There is no probabilistic model *per se*.

But some loss functions have simple and interesting properties, e.g. $\ell_s(x,y) = \mathbf{1}_{\mathbf{1}_{x>s}\neq y}$ for classification, associated with missclassification dummy (with cutoff threshold s),

	$y_i = 0$	$y_i = 1$
$m(\boldsymbol{x}_i) \leq s$	n_{00}	n_{01}
$m(\boldsymbol{x}_i) > s$	n_{10}	n_{11}

$$\widehat{m}_s(\cdot) = \underset{m(\cdot)\in\mathcal{F}}{\text{argmin}} \{n_{01} + n_{01}\}$$

OLS Regression, ℓ_2 norm and Expected Value: $\ell_2(x,y)=[x-y]^2$

Let $\boldsymbol{y}\in\mathbb{R}^n$, $\overline{y}=\underset{m\in\mathbb{R}}{\text{argmin}}\left\{\sum_{i=1}^n\frac{1}{n}\big[\underbrace{y_i-m}_{\varepsilon_i}\big]^2\right\}$. It is the empirical version of

$$\mathbb{E}[Y]=\underset{m\in\mathbb{R}}{\text{argmin}}\left\{\int\big[\underbrace{y-m}_{\varepsilon}\big]^2 dF(y)\right\}=\underset{m\in\mathbb{R}}{\text{argmin}}\left\{\mathbb{E}\Big[\|\underbrace{Y-m}_{\varepsilon}\|_{\ell_2}\Big]\right\}$$

where Y is a random variable.

Thus, $\underset{m(\cdot):\mathbb{R}^k\to\mathbb{R}}{\text{argmin}}\left\{\sum_{i=1}^n\frac{1}{n}\big[\underbrace{y_i-m(\boldsymbol{x}_i)}_{\varepsilon_i}\big]^2\right\}$ is the empirical version of $\mathbb{E}[Y|\boldsymbol{X}=\boldsymbol{x}]$.

See Legendre (1805) Nouvelles méthodes pour la détermination des orbites des comètes and Gauβ (1809) Theoria motus corporum coelestium in sectionibus conicis solem ambientium.

Median Regression, ℓ_1 norm and Median: $\ell_1(x,y)=|x-y|$

Let $\boldsymbol{y}\in\mathbb{R}^n$, $\text{median}[\boldsymbol{y}]\in\underset{m\in\mathbb{R}}{\text{argmin}}\left\{\sum_{i=1}^n\frac{1}{n}\underbrace{|y_i-m|}_{\varepsilon_i}\right\}$. It is the empirical version of

$$\text{median}[Y]\in\underset{m\in\mathbb{R}}{\text{argmin}}\left\{\int\underbrace{|y-m|}_{\varepsilon}dF(y)\right\}=\underset{m\in\mathbb{R}}{\text{argmin}}\left\{\mathbb{E}\big[\|\underbrace{Y-m}_{\varepsilon}\|_{\ell_1}\big]\right\}$$

where Y is a random variable, $\mathbb{P}[Y\leq\text{median}[Y]]\geq\frac{1}{2}$ and $\mathbb{P}[Y\geq\text{median}[Y]]\geq\frac{1}{2}$.

$\underset{m(\cdot):\mathbb{R}^k\rightarrow\mathbb{R}}{\text{argmin}}\left\{\sum_{i=1}^n\frac{1}{n}\underbrace{|y_i-m(\boldsymbol{x}_i)|}_{\varepsilon_i}\right\}$ is the empirical version of $\text{median}[Y|\boldsymbol{X}=\boldsymbol{x}]$.

See Boscovich (1757) De Litteraria expeditione per pontificiam ditionem ad dimetiendos duos meridiani and Laplace (1793) Sur quelques points du système du monde.

Quantiles and Expectiles, $\ell(x,y)=\mathcal{R}(x-y)$

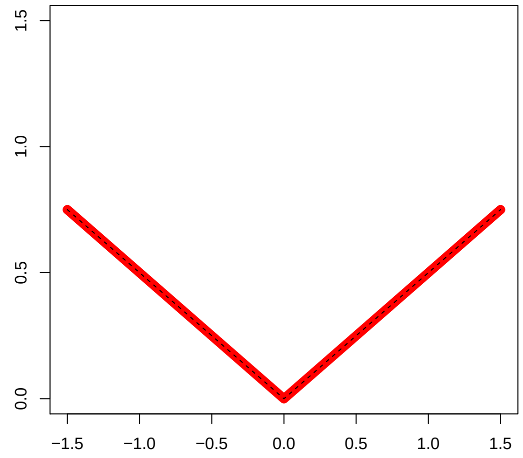
Consider the following risk functions

$$\mathcal{R}^{\mathsf{q}}_{\tau}(u)=u\cdot\big(\tau-\mathbf{1}(u<0)\big),\ \tau\in[0,1]$$

with $\mathcal{R}^{\mathsf{q}}_{1/2}(u)\propto|u|=\|u\|_{\ell_1}$, and

$$\mathcal{R}^{\mathsf{e}}_{\tau}(u)=u^2\cdot\big(\tau-\mathbf{1}(u<0)\big),\ \tau\in[0,1]$$

with $\mathcal{R}^{\mathsf{e}}_{1/2}(u)\propto u^2=\|u\|^2_{\ell_2}$.

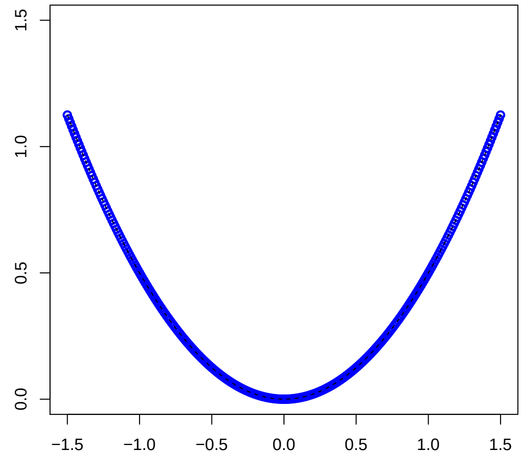


$$Q_Y(\tau)=\underset{m}{\text{argmin}}\big\{\mathbb{E}\big(\mathcal{R}^{\mathsf{q}}_{\tau}(Y-m)\big)\big\}$$

which is the median when $\tau=1/2$,

$$E_Y(\tau)=\underset{m}{\text{argmin}}\big\{\mathbb{E}\big(\mathcal{R}^{\mathsf{e}}_{\tau}(X-m)\big)\big\}$$

which is the expected value when $\tau=1/2$.



Elicitable Measures

"elicitable" means "being a minimizer of a suitable expected score"

T is an elicatable function if there exits a scoring function $S:\mathbb{R}\times\mathbb{R}\to[0,\infty)$ (see functions $\ell(\cdot,\cdot)$ introduced above) such that

$$T(Y)=\underset{x\in\mathbb{R}}{\text{argmin}}\left\{\int_{\mathbb{R}}S(x,y)dF(y)\right\}=\underset{x\in\mathbb{R}}{\text{argmin}}\left\{\mathbb{E}\big[S(x,Y)\big]\text{ where }Y\sim F.\right\}$$

see Gneiting (2011) Making and evaluating point forecasts.

The mean, $T(Y)=\mathbb{E}[Y]$ is elicited by $S(x,y)=\|x-y\|_{\ell_2}^2$

The median, $T(Y)=\text{median}[Y]$ is elicited by $S(x,y)=\|x-y\|_{\ell_1}$

The quantile, $T(Y)=Q_Y(\tau)$ is elicited by $S(x,y)=\tau(y-x)_+ + (1-\tau)(y-x)_-$

The expectile, $T(Y)=E_Y(\tau)$ is elicited by $S(x,y)=\tau(y-x)_+^2 + (1-\tau)(y-x)_-^2$

Quantiles and Expectiles (Technical Issues)

One can also write empirical quantiles and expectiles as

$$\text{quantile: argmin}\left\{\sum_{i=1}^n \omega_\tau^{\mathsf{q}}(\varepsilon_i)\Big|\underbrace{y_i-q_i}_{\varepsilon_i}\Big|\right\} \text{ where } \omega_\tau^{\mathsf{q}}(\epsilon)=\begin{cases}1-\tau \text{ if } \epsilon\leq 0\\ \tau \text{ if } \epsilon>0\end{cases}$$

$$\text{expectile: argmin}\left\{\sum_{i=1}^n \omega_\tau^{\mathsf{e}}(\varepsilon_i)\Big(\underbrace{y_i-q_i}_{\varepsilon_i}\Big)^2\right\} \text{ where } \omega_\tau^{\mathsf{e}}(\epsilon)=\begin{cases}1-\tau \text{ if } \epsilon\leq 0\\ \tau \text{ if } \epsilon>0\end{cases}$$

Expectiles are unique, not quantiles...

Quantiles satisfy $\mathbb{E}[\text{sign}(Y-Q_Y(\tau))]=0$

Expectiles satisfy $\tau\mathbb{E}\big[(Y-E_Y(\tau))_+\big]=(1-\tau)\mathbb{E}\big[(Y-E_Y(\tau))_-\big]$

(those are actually the first order conditions of the optimization problem).

Expectiles as Quantiles (Interpretation)

For every $Y \in L^1$, $\tau \mapsto E_Y(\tau)$ is continuous, and striclty increasing

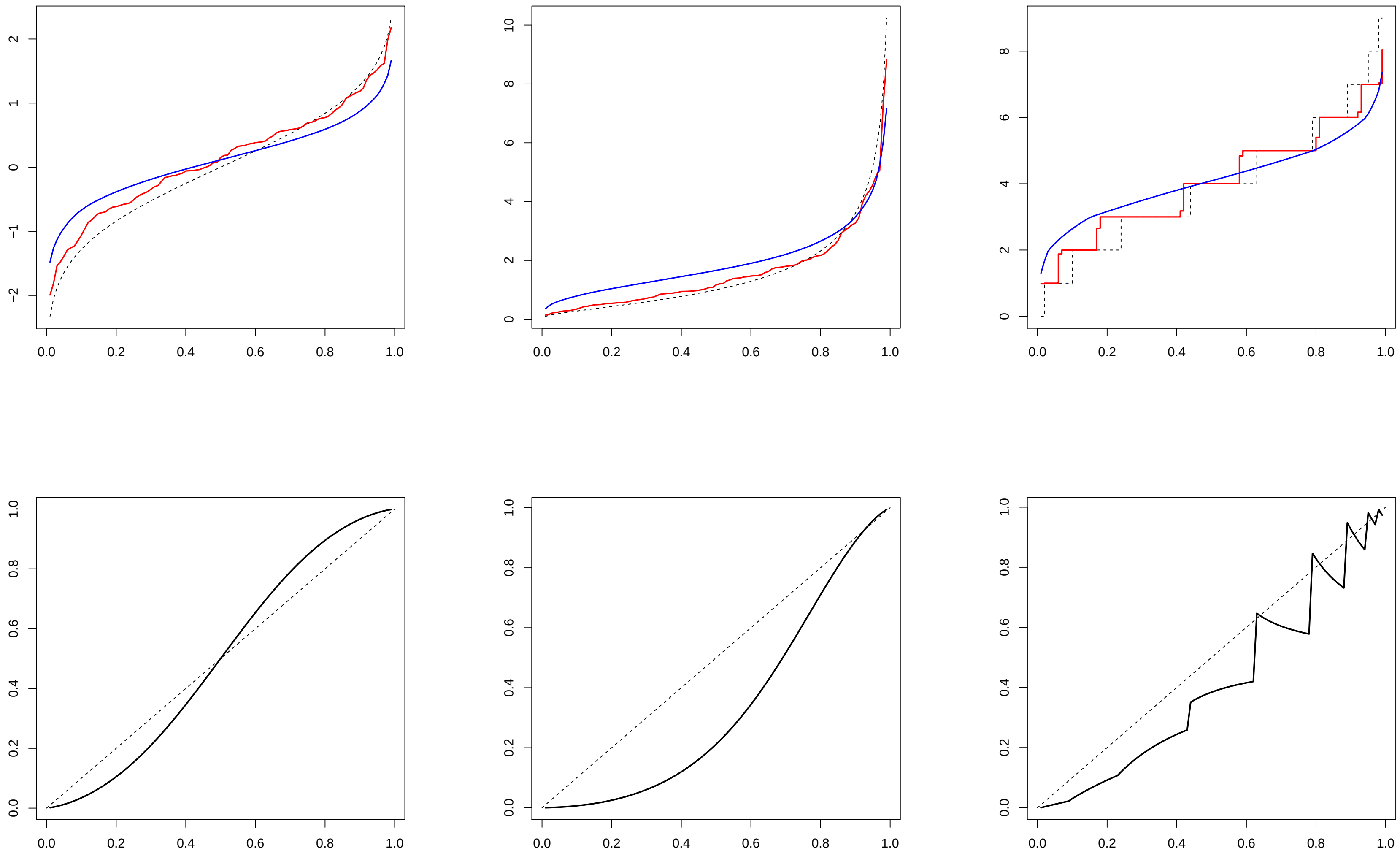
if Y is absolutely continuous, $\dfrac{\partial E_Y(\tau)}{\partial \tau} = \dfrac{\mathbb{E}[|X - E_Y(\tau)|]}{(1-\tau)F_Y(E_Y(\tau)) + \tau(1 - F_Y(E_Y(\tau)))}$

"*Expectiles have properties that are similar to quantiles*" Newey & Powell (1987) Asymmetric Least Squares Estimation and Testing. The reason is that expectiles of a distribution F are quantiles a distribution G which is related to F, see Jones (1994) Expectiles and M-quantiles are quantiles: let

$$G(t) = \frac{P(t) - tF(t)}{2[P(t) - tF(t)] + t - \mu} \text{ where } P(s) = \int_{-\infty}^{s} y dF(y).$$

The expectiles of F are the quantiles of G.

Distortion: Expectiles as Quantiles (Gaussian, Lognormal & Poisson)



Empirical Quantiles & Empirical Expectiles

Consider some i.id. sample $\{y_1, \cdots, y_n\}$ with distribution F (abs. continuous).

$$Q_\tau = \text{argmin}\left\{\mathbb{E}\left[\mathcal{R}^{\mathsf{q}}_\tau(Y-q)\right]\right\} \text{ where } Y \sim F \text{ and } \widehat{Q}_\tau \in \text{argmin}\left\{\sum_{i=1}^n \mathcal{R}^{\mathsf{q}}_\tau(y_i - q)\right\}$$

Then as $n \to \infty$, $\sqrt{n}\big(\widehat{Q}_\tau - Q_\tau\big) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \dfrac{\tau(1-\tau)}{f^2(Q_\tau)}\right)$

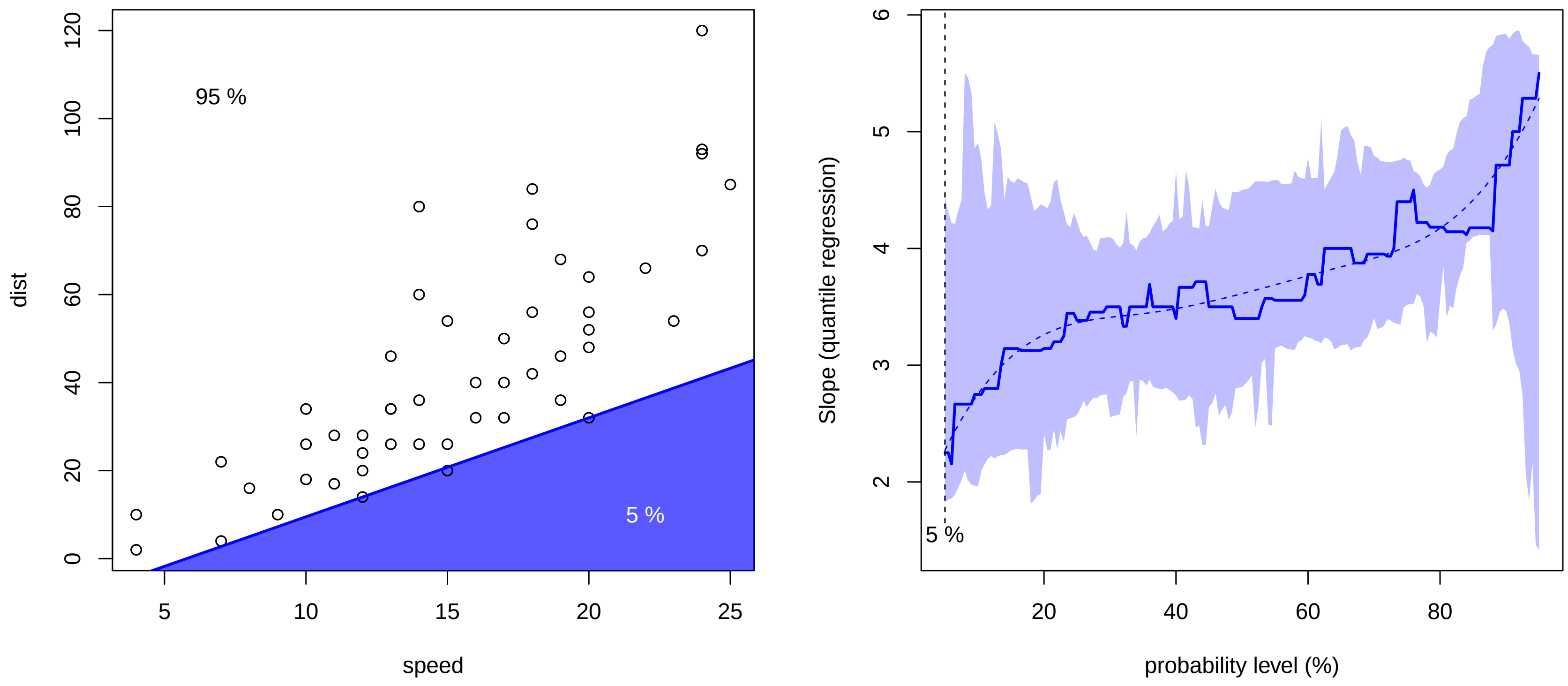
$$E_\tau = \text{argmin}\left\{\mathbb{E}\left[\mathcal{R}^{\mathsf{e}}_\tau(Y-m)\right]\right\} \text{ where } Y \sim F \text{ and } \widehat{E}_\tau = \text{argmin}\left\{\sum_{i=1}^n \mathcal{R}^{\mathsf{e}}_\tau(y_i - m)\right\}$$

Then as $n \to \infty$, $\sqrt{n}\big(\widehat{E}_\tau - E_\tau\big) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, s_\tau^2\right)$ where, if
$\mathcal{I}_\tau(x,y) = \tau(y-x)_+ + (1-\tau)(y-x)_-$ (elicitable score for quantiles),

$$s_\tau^2 = \frac{\mathbb{E}[\mathcal{I}_\tau(E_\tau, Y)^2]}{(\tau[1-F(E_\tau)] + [1-\tau]F(Eu_\tau))^2}.$$

Quantile Regression (and Heteroskedasticity)

We want to solve, here, $\min\left\{\sum_{i=1}^{n}\mathcal{R}_{\tau}^{\mathsf{q}}(y_i-\boldsymbol{x}_i^{\mathsf{T}}\boldsymbol{\beta})\right\}$



$$y_i=\boldsymbol{x}_i^{\mathsf{T}}\boldsymbol{\beta}+\varepsilon_i \text{ so that } \widehat{Q}_{y|\boldsymbol{x}}(\tau)=\boldsymbol{x}^{\mathsf{T}}\widehat{\boldsymbol{\beta}}+F_{\varepsilon}^{-1}(\tau)$$

Optimization Algorithm : Quantile Regression

Simplex algorithm to solve this program. Primal problem is

$$\min_{\boldsymbol{\beta},\boldsymbol{u},\boldsymbol{v}} \left\{\tau\mathbf{1}^\top\boldsymbol{u} + (1-\tau)\mathbf{1}^\top\boldsymbol{v}\right\} \text{ s.t. } \boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u} - \boldsymbol{v}, \text{ with } \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}_+^n$$

and the dual version is

$$\max_{\boldsymbol{d}} \left\{\boldsymbol{y}^\top\boldsymbol{d}\right\} \text{ s.t. } \boldsymbol{X}^\top\boldsymbol{d} = (1-\tau)\boldsymbol{X}^\top\mathbf{1} \text{ with } \boldsymbol{d} \in [0,1]^n$$

Koenker & D'Orey (1994) A Remark on Algorithm AS 229: Computing Dual Regression Quantiles and Regression Rank Scores suggest to use the simplex method (default method in R). Portnoy & Koenker (1997) The Gaussian hare and the Laplacian tortoise suggest to use the interior point method.

Running time is of order $n^{1+\delta}k^3$ for some $\delta > 0$ and $k = \dim(\boldsymbol{\beta})$

(it is $(n+k)k^2$ for OLS, see wikipedia).

Quantile Regression Estimators

OLS estimator $\widehat{\beta}^{\text{ols}}$ is solution of

$$\widehat{\beta}^{\text{ols}} = \text{argmin}\left\{\mathbb{E}\Big[\big(\mathbb{E}[Y|\boldsymbol{X}=\boldsymbol{x}] - \boldsymbol{x}^{\mathsf{T}}\boldsymbol{\beta}\big)^2\Big]\right\}$$

and Angrist, Chernozhukov & Fernandez-Val (2006) Quantile Regression under Misspecification proved that

$$\widehat{\beta}_{\tau}^{\text{q}} = \text{argmin}\left\{\mathbb{E}\Big[\omega_{\tau}(\boldsymbol{\beta})\big(Q_{\tau}[Y|\boldsymbol{X}=\boldsymbol{x}] - \boldsymbol{x}^{\mathsf{T}}\boldsymbol{\beta}\big)^2\Big]\right\}$$

(under weak conditions) where

$$\omega_{\tau}(\boldsymbol{\beta}) = \int_0^1 (1-u) f_{y|\boldsymbol{x}}(u\boldsymbol{x}^{\mathsf{T}}\boldsymbol{\beta} + (1-u)Q_{\tau}[Y|\boldsymbol{X}=\boldsymbol{x}])du$$

$\widehat{\beta}_{\tau}^{\text{q}}$ is the best weighted mean square approximation of the true quantile function, where the weights depend on an average of the conditional density of Y over $\boldsymbol{x}^{\mathsf{T}}\boldsymbol{\beta}$ and the true quantile regression function.

Quantile Regression Estimators

Under weak conditions, $\widehat{\boldsymbol{\beta}}_\tau^{\mathsf{q}}$ is asymptotically normal:

$$\sqrt{n}(\widehat{\boldsymbol{\beta}}_\tau^{\mathsf{q}} - \boldsymbol{\beta}_\tau^{\mathsf{q}}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \tau(1-\tau)D_\tau^{-1}\Omega_x D_\tau^{-1}),$$

where

$$D_\tau = \mathbb{E}\big[f_\varepsilon(0)\boldsymbol{X}\boldsymbol{X}^{\mathsf{T}}\big] \text{ and } \Omega_x = \mathbb{E}\big[\boldsymbol{X}^{\mathsf{T}}\boldsymbol{X}\big].$$

hence, the asymptotic variance of $\widehat{\boldsymbol{\beta}}$ is

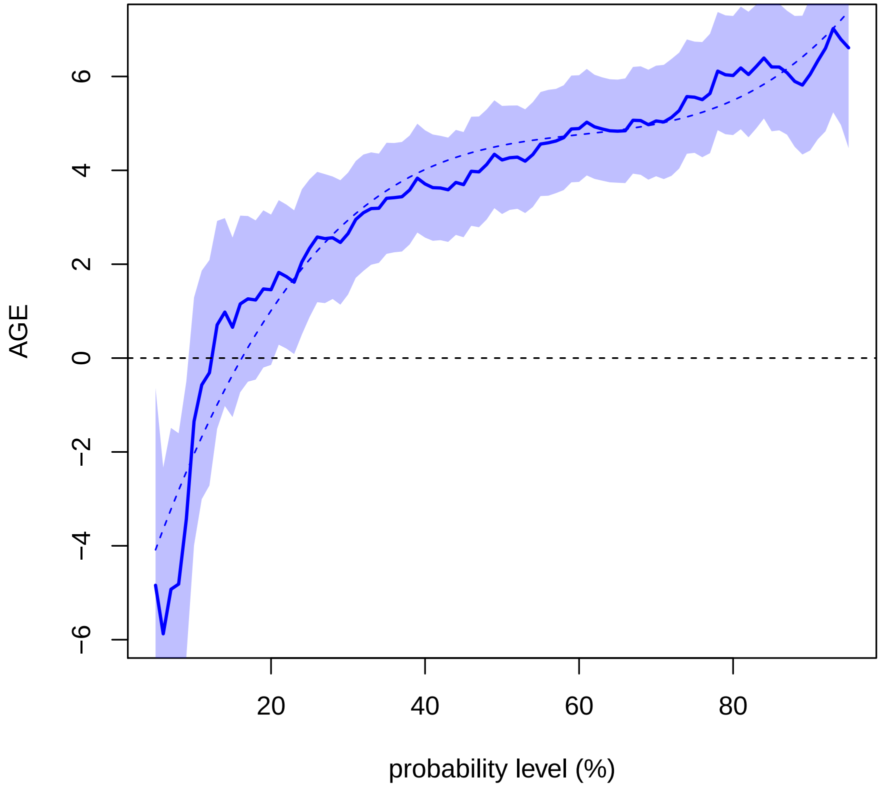
$$\widehat{\text{Var}}\big[\widehat{\boldsymbol{\beta}}_\tau^{\mathsf{q}}\big] = \frac{\tau(1-\tau)}{[\widehat{f}_\varepsilon(0)]^2}\left(\frac{1}{n}\sum_{i=1}^n \boldsymbol{x}_i^{\mathsf{T}}\boldsymbol{x}_i\right)^{-1}$$

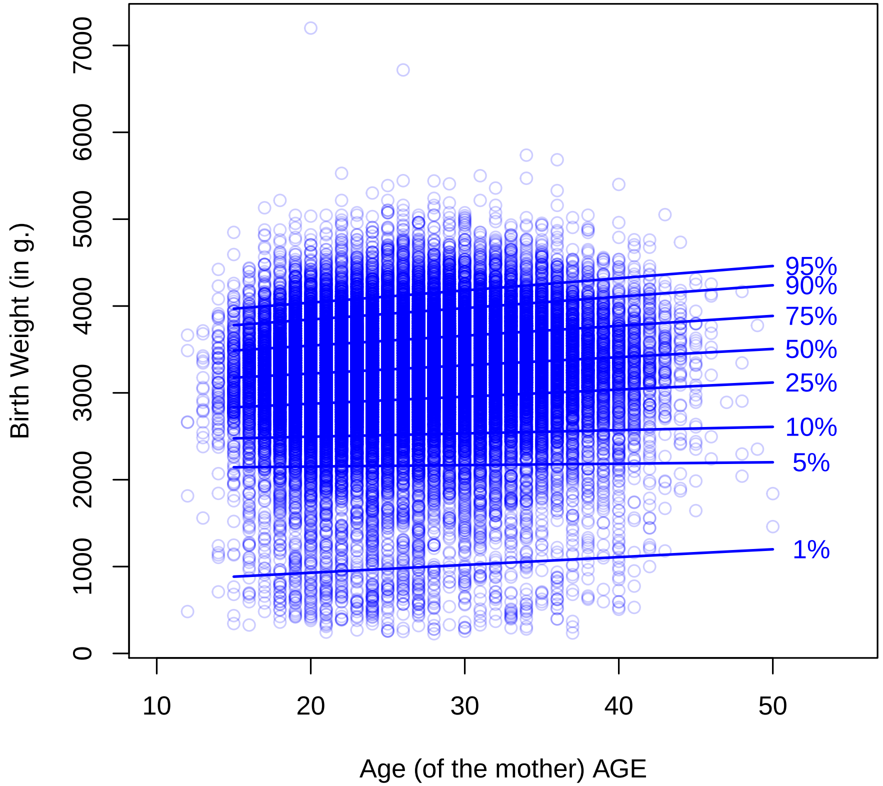
where $\widehat{f}_\varepsilon(0)$ is estimated using (e.g.) an histogram, as suggested in Powell (1991) Estimation of monotonic regression models under quantile restrictions, since

$$D_\tau = \lim_{h\downarrow 0}\mathbb{E}\left(\frac{\mathbf{1}(|\varepsilon|\leq h)}{2h}\boldsymbol{X}\boldsymbol{X}^{\mathsf{T}}\right) \sim \frac{1}{2nh}\sum_{i=1}^n \mathbf{1}(|\varepsilon_i|\leq h)\boldsymbol{x}_i\boldsymbol{x}_i^{\mathsf{T}} = \widehat{D}_\tau.$$

Visualization, $\tau \mapsto \widehat{\beta}^{\mathbf{q}}_{\tau}$

See Abreveya (2001) The effects of demographics and maternal behavior on the distribution of birth outcomes





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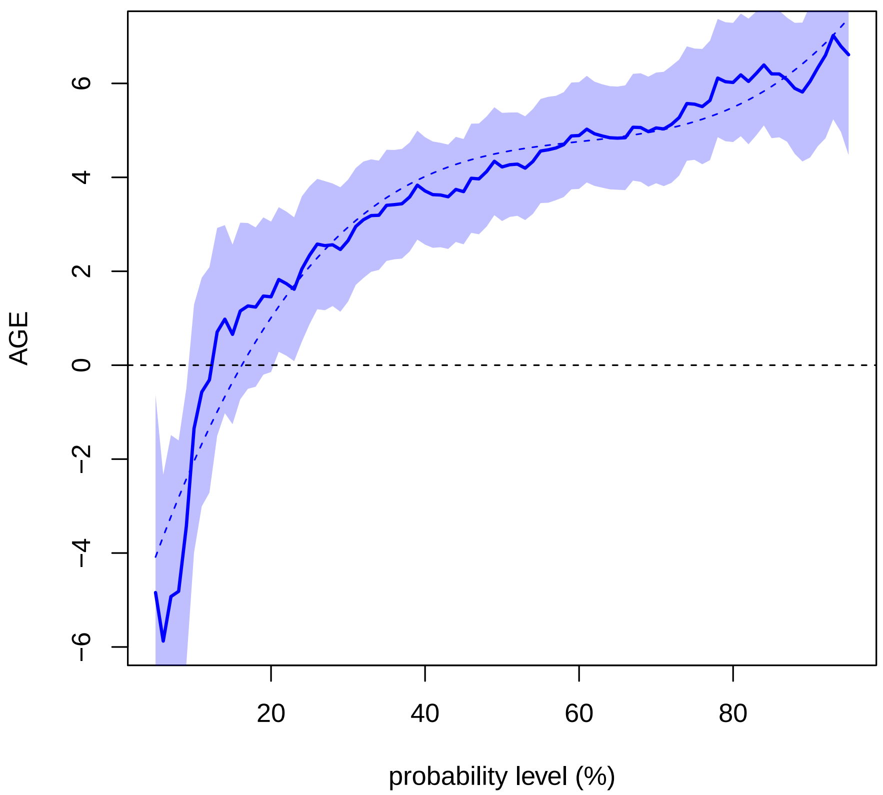
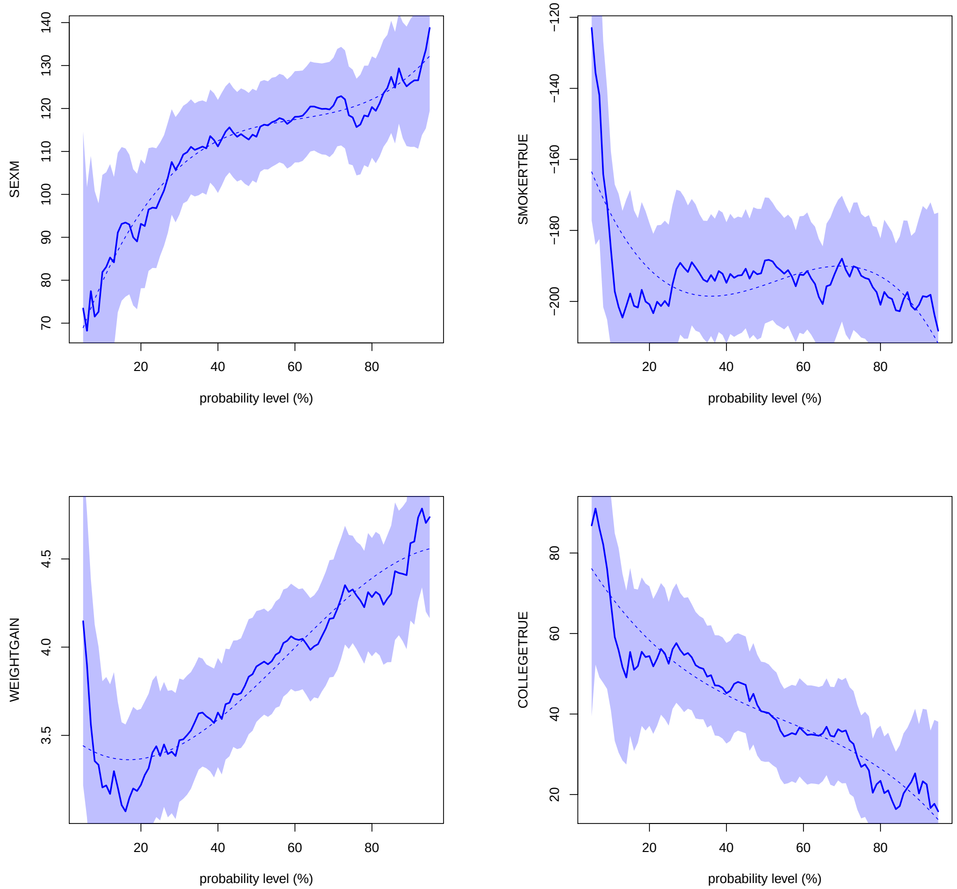




Quantile Regression on Panel Data

In the context of panel data, consider some fixed effect, α_i so that

$$y_{i,t} = \boldsymbol{x}_{i,t}^\mathsf{T}\boldsymbol{\beta}_\tau + \alpha_i + \varepsilon_{i,t} \text{ where } Q_\tau(\varepsilon_{i,t}|\boldsymbol{X}_i) = 0$$

Canay (2011) A simple approach to quantile regression for panel data suggests an estimator in two steps,

- use a standard OLS fixed-effect model $y_{i,t} = \boldsymbol{x}_{i,t}^\mathsf{T}\boldsymbol{\beta} + \alpha_i + u_{i,t}$, i.e. consider a within transformation, and derive the fixed effect estimate $\widehat{\boldsymbol{\beta}}$

$$(y_{i,t} - \overline{y}_i) = \big(\boldsymbol{x}_{i,t} - \overline{\boldsymbol{x}}_{i,t}\big)^\mathsf{T}\boldsymbol{\beta} + (u_{i,t} - \overline{u}_i)$$

- estimate fixed effects as $\widehat{\alpha}_i = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T}\big(y_{i,t} - \boldsymbol{x}_{i,t}^\mathsf{T}\widehat{\boldsymbol{\beta}}\big)$
- finally, run a standard quantile regression of $y_{i,t} - \widehat{\alpha}_i$ on $\boldsymbol{x}_{i,t}$'s.

See rqpd package.

Quantile Regression with Fixed Effects (QRFE)

In a panel linear regression model, $y_{i,t} = \boldsymbol{x}_{i,t}^\mathsf{T}\boldsymbol{\beta} + u_i + \varepsilon_{i,t}$,

where u is an unobserved individual specific effect.

In a fixed effects models, u is treated as a parameter. Quantile Regression is

$$\min_{\boldsymbol{\beta},\boldsymbol{u}} \left\{ \sum_{i,t} \mathcal{R}_\tau^\mathsf{q}(y_{i,t} - [\boldsymbol{x}_{i,t}^\mathsf{T}\boldsymbol{\beta} + u_i]) \right\}$$

Consider Penalized QRFE, as in Koenker & Bilias (2001) Quantile regression for duration data,

$$\min_{\boldsymbol{\beta}_1,\cdots,\boldsymbol{\beta}_\kappa,\boldsymbol{u}} \left\{ \sum_{k,i,t} \omega_k \mathcal{R}_{\tau_k}^\mathsf{q}(y_{i,t} - [\boldsymbol{x}_{i,t}^\mathsf{T}\boldsymbol{\beta}_k + u_i]) + \lambda \sum_i |u_i| \right\}$$

where ω_k is a relative weight associated with quantile of level τ_k.

Quantile Regression with Random Effects (QRRE)

Assume here that $y_{i,t} = \boldsymbol{x}_{i,t}^{\mathsf{T}}\boldsymbol{\beta} + \underbrace{u_i + \varepsilon_{i,t}}_{=\eta_{i,t}}$.

Quantile Regression Random Effect (QRRE) yields solving

$$\min_{\boldsymbol{\beta}} \left\{ \sum_{i,t} \mathcal{R}_\tau^{\mathsf{q}}(y_{i,t} - \boldsymbol{x}_{i,t}^{\mathsf{T}}\boldsymbol{\beta}) \right\}$$

which is a weighted asymmetric least square deviation estimator.

Let $\Sigma = [\sigma_{s,t}(\tau)]$ denote the matrix

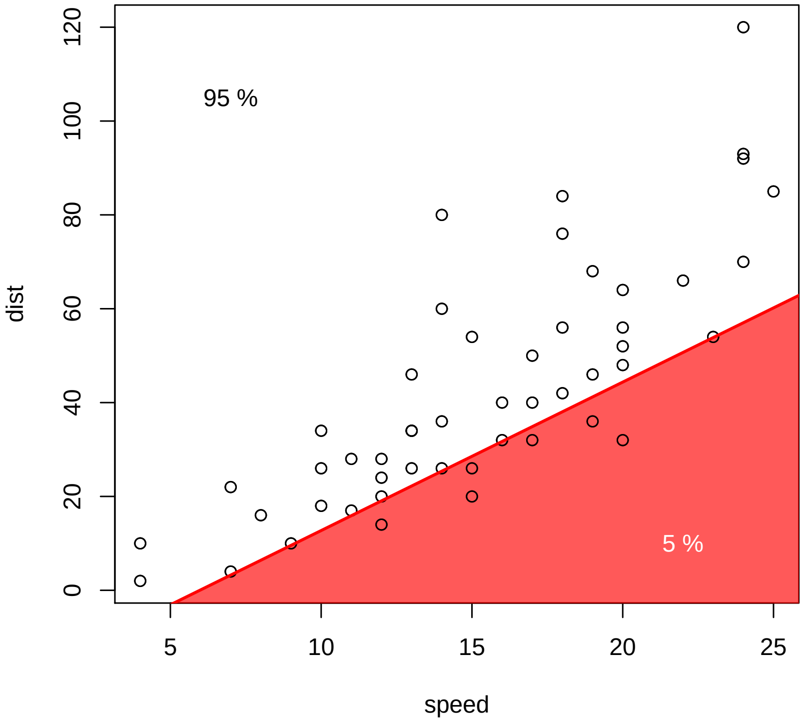
$$\sigma_{ts}(\tau) = \begin{cases} \tau(1-\tau) & \text{if } t = s \\ \mathbb{E}[\mathbf{1}\{\varepsilon_{it}(\tau) < 0, \varepsilon_{is}(\tau) < 0\}] - \alpha^2 & \text{if } t \neq s \end{cases}$$

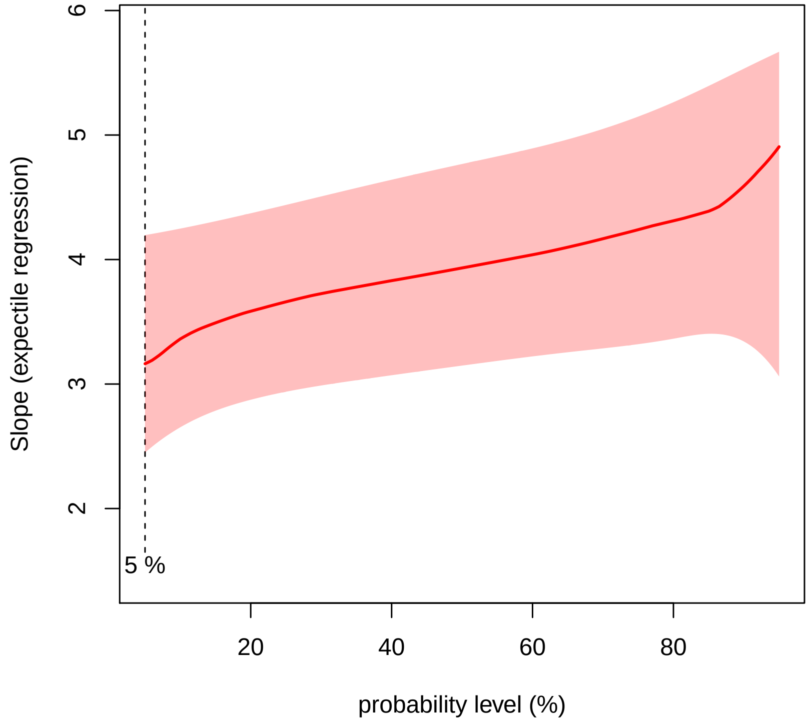
If $(nT)^{-1}\boldsymbol{X}^{\mathsf{T}}\{\mathbb{I}_n \otimes \Sigma_{T\times T}(\tau)\}\boldsymbol{X} \to \mathbf{D}_0$ as $n \to \infty$ and $(nT)^{-1}\boldsymbol{X}^{\mathsf{T}}\Omega_f\boldsymbol{X} = \mathbf{D}_1$, then

$$\sqrt{nT}\left(\widehat{\boldsymbol{\beta}}_\tau^{\mathsf{q}} - \boldsymbol{\beta}_\tau^{\mathsf{q}}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \mathbf{D}_1^{-1}\mathbf{D}_0\mathbf{D}_1^{-1}\right).$$

Expectile Regression

We want to solve, here, $\min_{\beta}\left\{\sum_{i=1}^{n}\mathcal{R}_{\tau}^{\mathsf{e}}(y_i-\boldsymbol{x}_i^{\mathsf{T}}\beta)\right\}$





see Koenker (2014) Living Beyond our Means for a comparison quantiles-expectiles

Expectile Regression

Solve here $\min_{\boldsymbol{\beta}}\left\{\sum_{i=1}^{n}\mathcal{R}_{\tau}^{\mathsf{e}}(y_i-\boldsymbol{x}_i^{\mathsf{T}}\boldsymbol{\beta})\right\}$ where $\mathcal{R}_{\tau}^{\mathsf{e}}(u)=u^2\cdot\big(\tau-\mathbf{1}(u<0)\big)$

"*this estimator can be interpreted as a maximum likelihood estimator when the disturbances arise from a normal distribution with unequal weight placed on positive and negative disturbances*" Aigner, Amemiya & Poirier (1976) Formulation and Estimation of Stochastic Frontier Production Function Models.

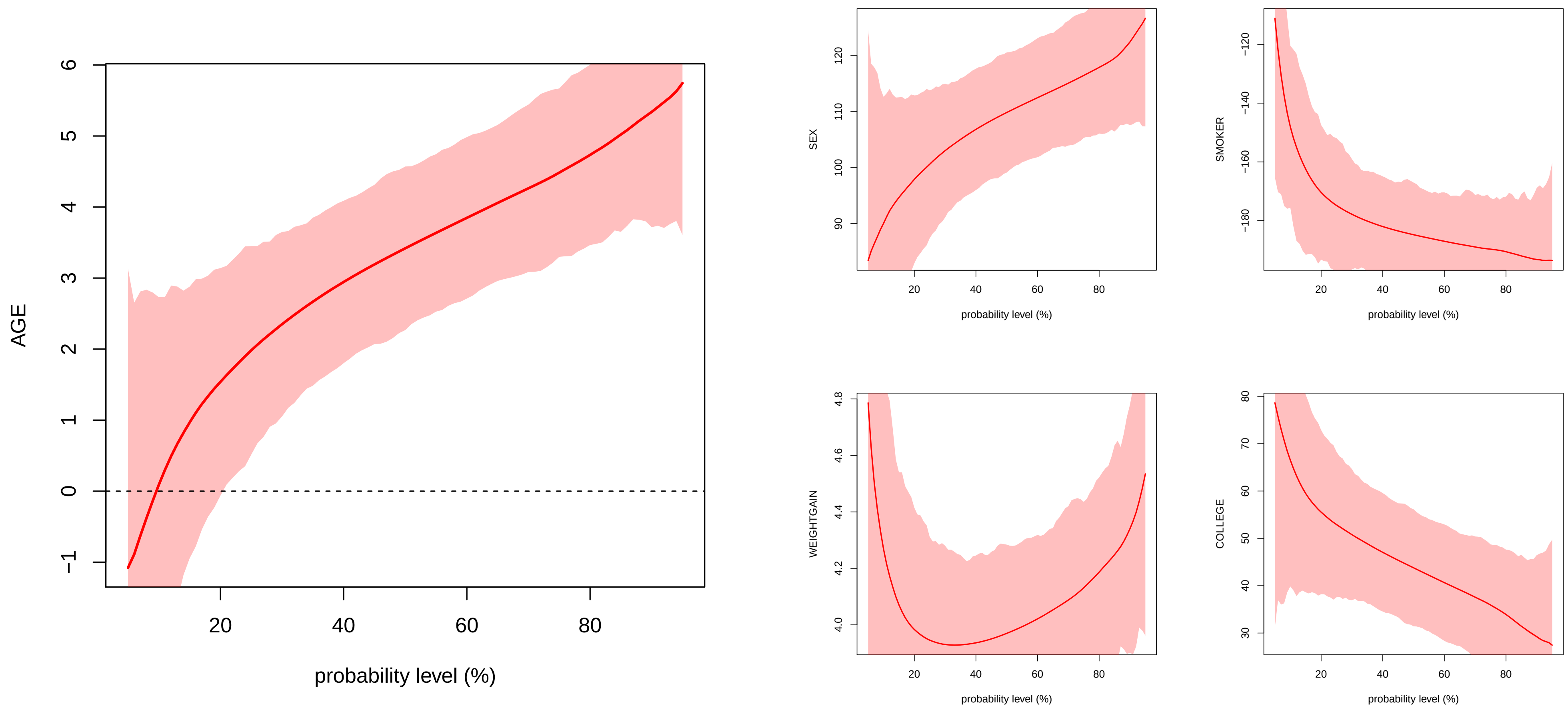
See Holzmann & Klar (2016) Expectile Asymptotics for statistical properties.

Expectiles can (also) be related to Breckling & Chambers (1988) M-Quantiles.

Comparison quantile regression and expectile regression, see Schulze-Waltrup *et al.* (2014) Expectile and quantile regression - David and Goliath?

Visualization, $\tau \mapsto \widehat{\beta}^{\mathbf{e}}_{\tau}$

See Abreveya (2001) The effects of demographics and maternal behavior on the distribution of birth outcomes



Expectile Regression, with Random Effects (ERRE)

Quantile Regression Random Effect (QRRE) yields solving

$$\min_{\boldsymbol{\beta}}\left\{\sum_{i,t}\mathcal{R}^{\mathsf{e}}_{\alpha}(y_{i,t}-\boldsymbol{x}_{i,t}^{\mathsf{T}}\boldsymbol{\beta})\right\}$$

One can prove that

$$\widehat{\boldsymbol{\beta}}^{\mathsf{e}}_{\tau}=\left(\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{\omega}_{i,t}(\tau)\boldsymbol{x}_{it}\boldsymbol{x}_{it}^{\mathsf{T}}\right)^{-1}\left(\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{\omega}_{i,t}(\tau)\boldsymbol{x}_{it}y_{it}\right),$$

where $\widehat{\omega}_{it}(\tau)=\left|\tau-\mathbf{1}(y_{it}<\boldsymbol{x}_{it}^{\mathsf{T}}\widehat{\boldsymbol{\beta}}^{\mathsf{e}}_{\tau})\right|$.

Expectile Regression with Random Effects (ERRE)

If $W = \text{diag}(\omega_{11}(\tau), \dots \omega_{nT}(\tau))$, set

$$\overline{W} = \mathbb{E}(W), H = \boldsymbol{X}^\mathsf{T}\overline{W}\boldsymbol{X} \text{ and } \Sigma = \boldsymbol{X}^\mathsf{T}\mathbb{E}(W\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^\mathsf{T}W)\boldsymbol{X}.$$

and then

$$\sqrt{nT}\{\widehat{\boldsymbol{\beta}}_\tau^{\mathsf{e}} - \boldsymbol{\beta}_\tau^{\mathsf{e}})\} \xrightarrow{\mathcal{L}} \mathcal{N}(0, H^{-1}\Sigma H^{-1}),$$

see Barry *et al.* (2016) Quantile and Expectile Regression for random effects model.

Application to Real Data